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This is part of a student text which was written with the aim of reflecting the thinking of The Cambridge Conference on School Mathematics (CCSM) regarding the goals and objectives for mathematics. The instructional materials were developed for teaching geometry in the secondary schools. This document is chapter six and titled Motions and Transformations. Presented is the concept of rigid motion in the plane. Various kinds of rigid motions are considered, certain mathematical ideas about rigid motions are obtained, and a number of applications are described. One of the chief mathematical ideas presented is that every rigid motion can be viewed either as a translation, a rotation, a reflection, or a combination of reflection and translation. This idea and others lead to a variety of useful applications in geometry. Several of these applications involving rigid motions are used to solve geometrical problems. Both explanatory materials and student problems are included. [Not available in hard copy due to marginal legibility of original document]. (RP)

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CCSM
Feasibility Study No. 43a

Student
Text

AN EXPERIMENTAL TEXT IN TRANSFORMATIONAL GEOMETRY

Educational Services Incorporated
Newton, Massachusetts

SE 007 213

Prepared at the 1965
Mombasa Mathematics Workshop

Issued by
Cambridge Conference on School Mathematics
Educational Services Incorporated

CONTENTS

Chapter 6

MOTIONS AND TRANSFORMATIONS

6-1 FIGURES WITH THE SAME SIZE AND SHAPE.

Look at the following triangles. Do they have the same size and shape?

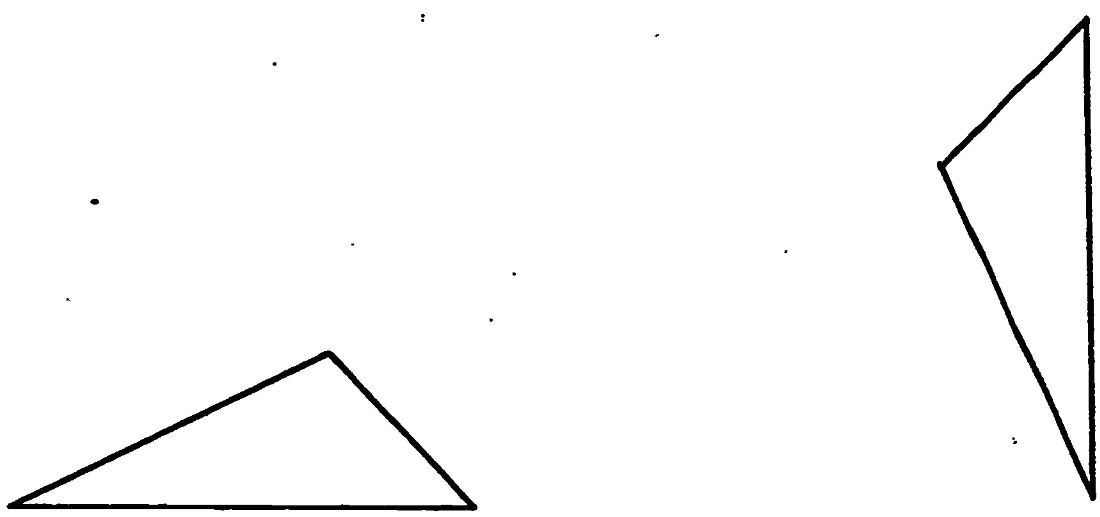

Fig. 1.

One way to test whether or not they have the same size and shape is to measure the sides of each triangle with a ruler and the angles of each triangle with a protractor. If you obtain the same measurements for each triangle, then, as far as you can tell by your measurements, the triangles do have the same size and shape.

In work on geometry that you have already done, you learned that if you measure the sides (but not the angles) of two triangles, this is enough.

If the measurements of the sides for one triangle are the same as the measurements of the sides for the other triangle, then the measurements of the angles for one triangle must also be the same as for the other triangle. Can you state the theorem which showed this?

Using dividers or a pair of compasses to compare sides, test whether the following two triangles have the same size and shape.

Fig. 2.

Look at the following circles. Do they have the same size and shape?

Fig. 3.

One way to test whether or not they have the same size and shape is first to find the centre of each circle and then to measure the radius of each circle. If the measurements are the same, then the two circles have the same size and shape.

Using compasses and a ruler, test whether or not the following two circles have the same size and shape.

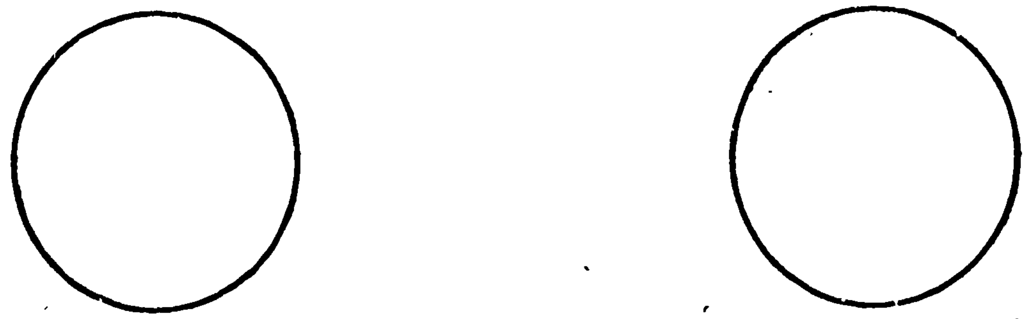

Fig. 4.

(Remember to use the construction for finding the centre of a circle which you learned in Chapter 4.)

In our earlier work, if two triangles had the same size and shape we said that the two triangles were *congruent*. If two circles had the same size and shape, we said that the two circles were *congruent*.

Now look at the following two figures. Do these figures have the same size and shape?

Fig. 5.

Clearly there are no simple measurements with ruler and compasses that you can use to answer this question. Think hard and decide what you would do if you had to get an answer.

One good way to get an answer is the following. Take a piece of tracing paper (any thin paper that you can see through a little bit will do), and place this paper over figure A. Then, holding this paper so that it does not move while you are writing on it, carefully trace, with your pencil, an outline of figure A. Next, move the paper over to figure B and try to find a position where the traced figure falls exactly on top of figure B. If you succeed, then you will know that, as far as you can tell by your drawings, the two figures do indeed have the same size and shape.

Do the above figures have the same size and shape?

CLASS ACTIVITY

For each of the following pairs of figures, test whether or not the two figures have the same size and shape.

a.

b.

c.

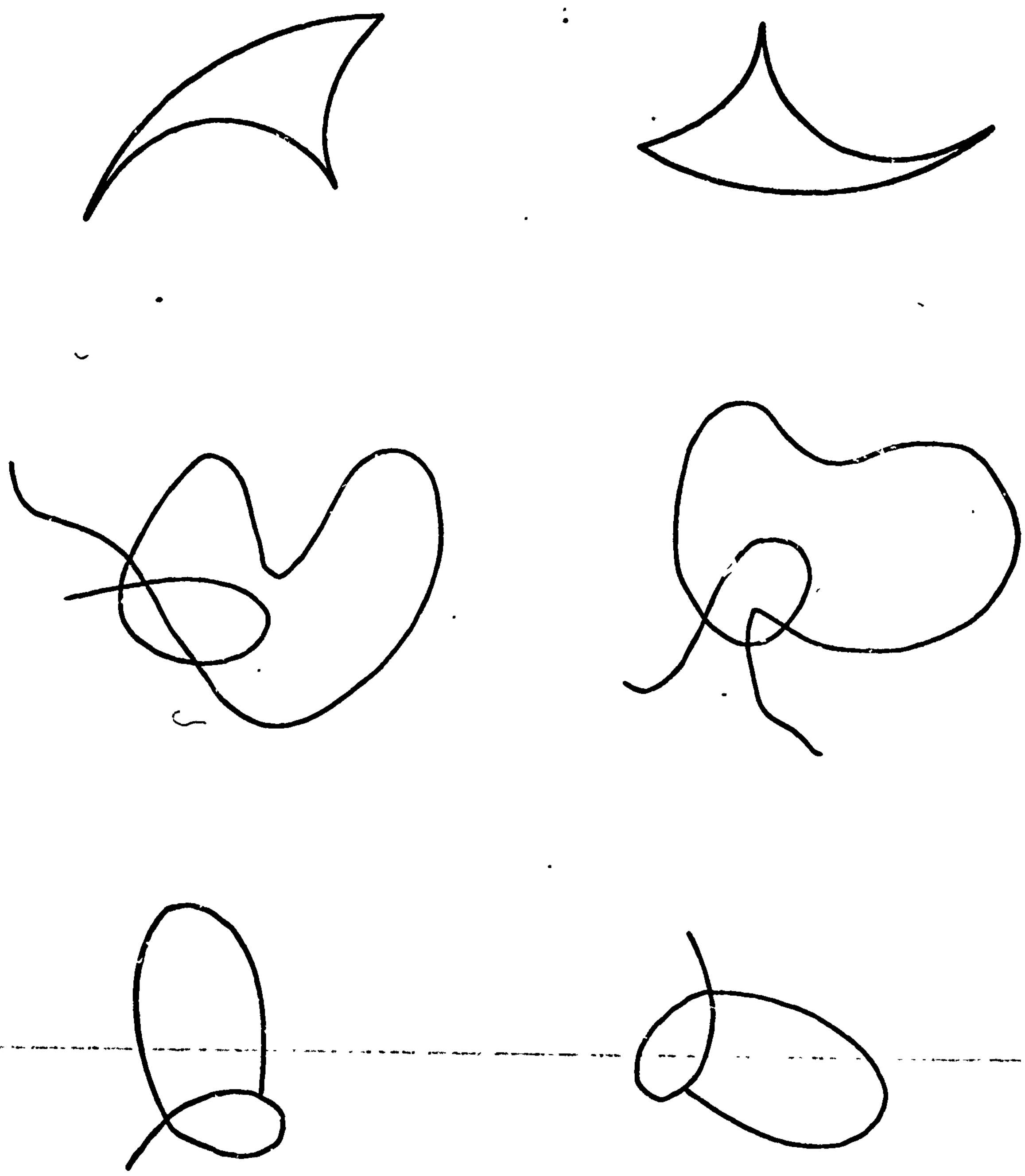

Fig. 6.

Consider the following pair of figures.

Fig. 7.

Make a tracing of figure A above. Now compare this tracing with figure B. You will find that you cannot make the tracing of A fall exactly on B. Now turn the tracing paper over so that the tracing of A is on the bottom. Place this over B and, looking through the paper, see if you can make the tracing of A fall on B. You will find that the tracing can now be made to fall exactly on B.

Shall we say that A and B have the same size and shape? In geometry, mathematicians long ago agreed to use the words "same size and shape" in such a case. We therefore do say that A and B have the same size and shape.

In general, we say that two figures in the plane have the *same size and shape* if a tracing of one can be made to fall exactly on the other, where we may turn the tracing over if needed.

PROBLEMS 6-1

1. Do the following figures have the same size and shape?

a.

b. 

c. 

Fig. 8.

2. For each of the figures in Figure 9, make a tracing of the first figure. For how many distinct positions of the tracing paper can you make the tracing fall exactly on the second figure? (Include positions where the tracing paper is turned over.)

a. 

b. 

c.

d.

e.

Fig. 9.

6-2 MOVING A FIGURE IN THE PLANE.

By using tracing paper, we can take a figure in the plane and make a copy of it at a new position in the plane. Here is an example.

Consider the following picture.

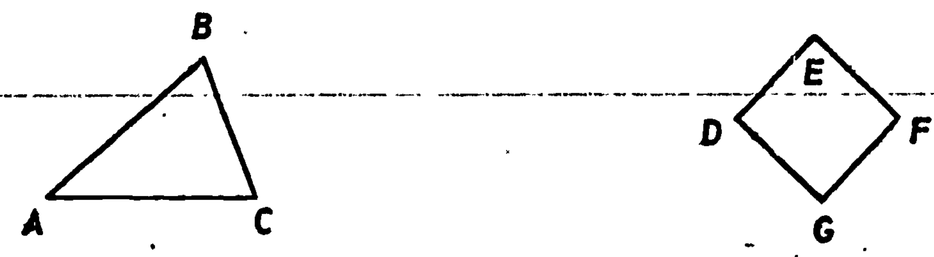

Fig. 10.

Make a copy of this picture on your own paper, and call the plane of the picture **P**. We are now going to make a copy of the triangle ABC at a new position in **P**. We shall place this copy of the triangle ABC so that point C falls on point F and line segment $\overline{AC}$ falls on the line of segment $\overline{EF}$. We do this as follows. Take a piece of tracing paper. Call this paper **T**. Place paper **T** on top of **P** and make a copy of triangle ABC on paper **T**. Next, move paper **T** so that point C of paper **T** falls above point F of plane **P**, and line segment $\overline{AC}$ of paper **T** falls along line segment $\overline{EF}$ of **P**. Keeping the paper in this position, copy triangle ABC from paper **T** onto **P**. This may be done by using a point of your compasses or of your dividers to prick through from paper **T** to **P** at points A, B, and C, of paper **T**, and then using a ruler to draw the line segments between points on **P**. Plane **P** will now look like this.

Fig. 11.

Here we have used A', B', and C' to name the vertices of the triangle in its new position.

When we have done all this, we say that we have *moved* the triangle to a new position *in the plane of* **P**. There are now two triangles in **P**: triangle ABC and triangle $A'B'C'$. In such a case, we shall often think of these as the *same* triangle in two different *positions*.

PROBLEMS 6-2A

1. Copy the following picture. In this picture, line L is parallel to $\overline{AD}$. Use tracing paper to move the rectangle $ABCD$ so that point D falls on point E of the circle and each side of the rectangle in its new position is parallel to its first position.

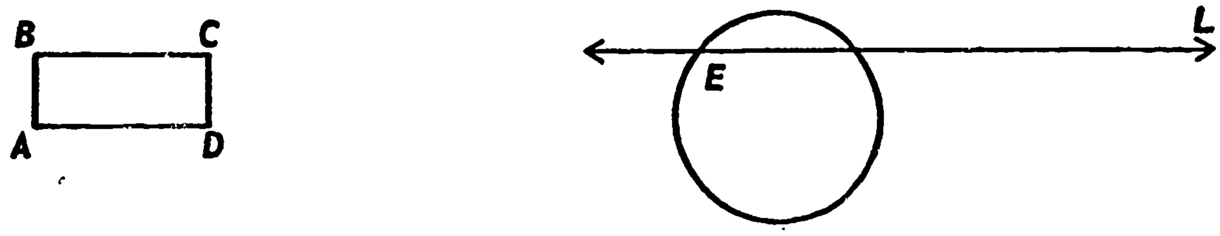

Fig. 12.

2. In the same way, in the following picture, move the triangle ABC so that the point C falls on its original position and the segment $\overline{CB}$ falls along the line of segment $\overline{CD}$.

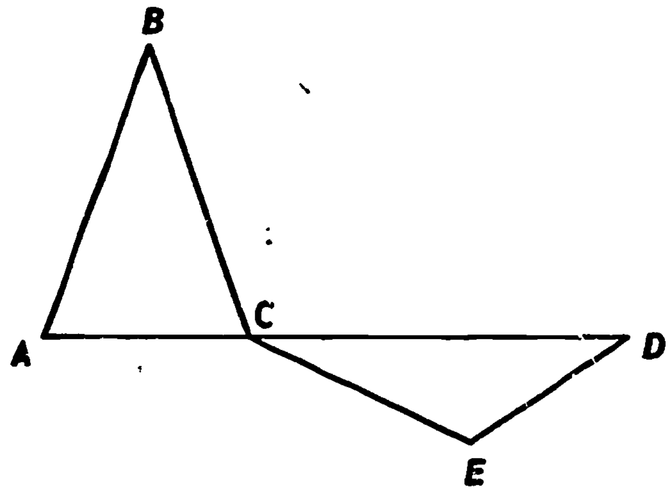

Fig. 13.

3. In the same way, in the following picture, move triangle ABC so that segment $\overline{AB}$ falls along segment $\overline{DE}$, point A falls on point F, and triangle ABC falls entirely inside the circle.

Fig. 14.

(Note that you will have to turn the tracing paper over before you copy the triangle back onto the picture.)

4. Show that Problem 1 has four different answers and that Problem 2 has four different answers.

Consider the following figure,

Fig. 15.

and the movement of the tracing paper which carries A to C and $\overline{AB}$ onto the line of $\overline{CD}$. This gives Figure 16 below.

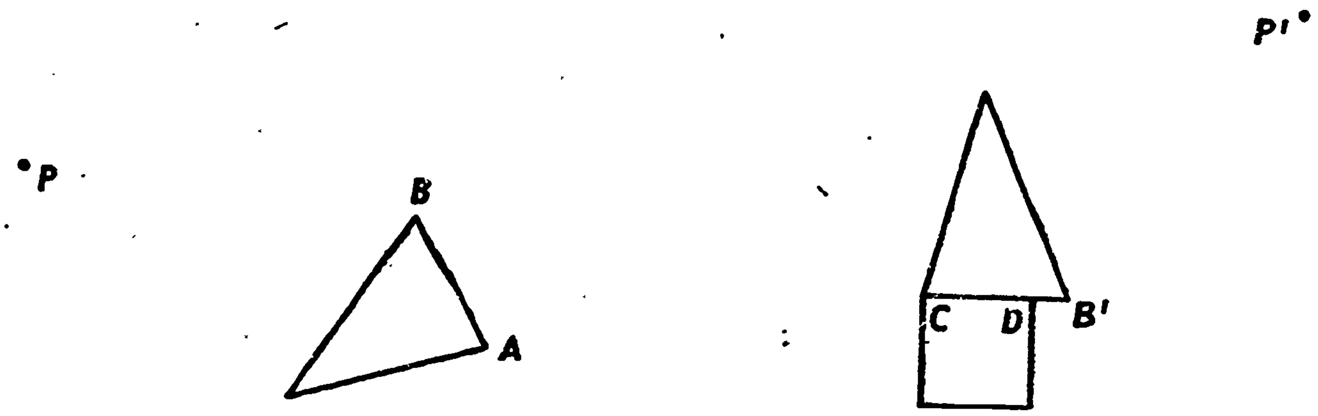

Fig. 16.

Now take any point in the original figure. Call it P. The same movement of the tracing paper which carries the triangle to the top of the square also carries the point P to a new position P'. We can get P' by marking P on the tracing paper at the same time that we copy the triangle and then pricking through, after we have moved the tracing paper.

The following picture shows the result for several different points P_1, P_2, P_3. P_1' is the new position of P_1; P_2' is the new position of P_2; and P_3' is the new position of P_3.

Note that in this movement of the tracing paper, some points may move farther than others.

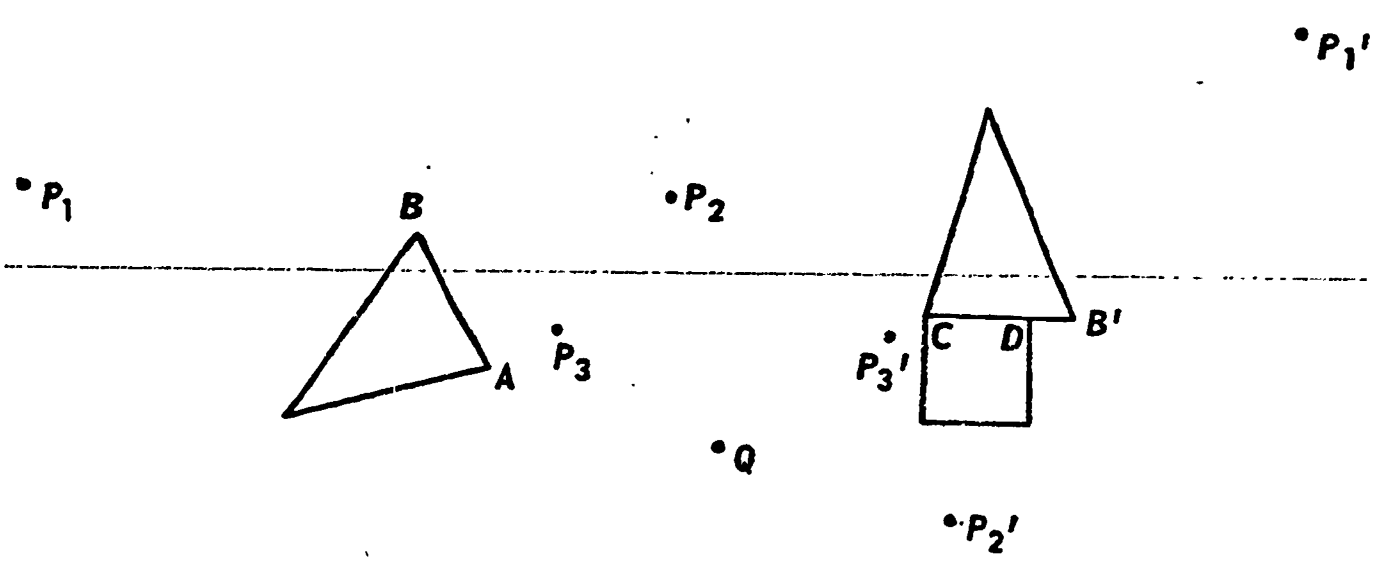

Fig. 17.

(Indeed, we can see that there is one point which does not get moved at all. Its new position is the same as its old position. This point is marked Q in Figure 17.)

Sometimes a movement of the tracing paper will move the individual points in a figure, but leave the final form and position of the entire figure unchanged. For example, if we add the following equilateral triangle to Figure 17,

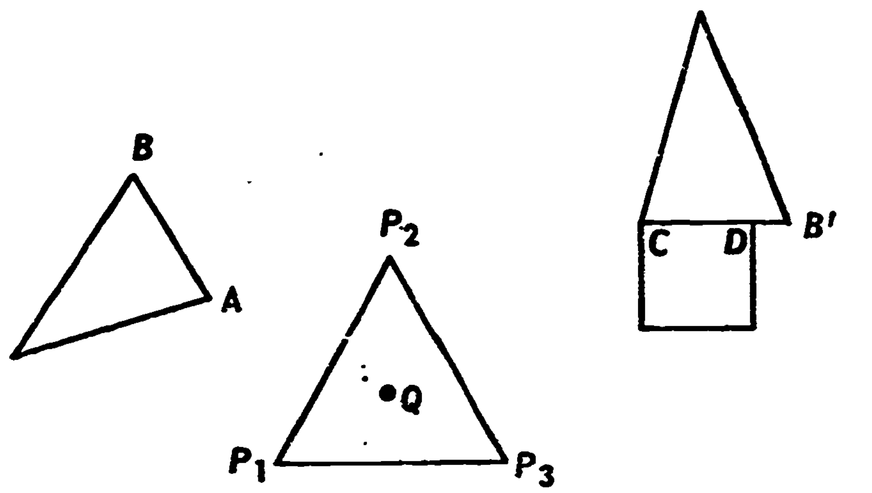

Fig. 18.

then the movement described before gives the following result.

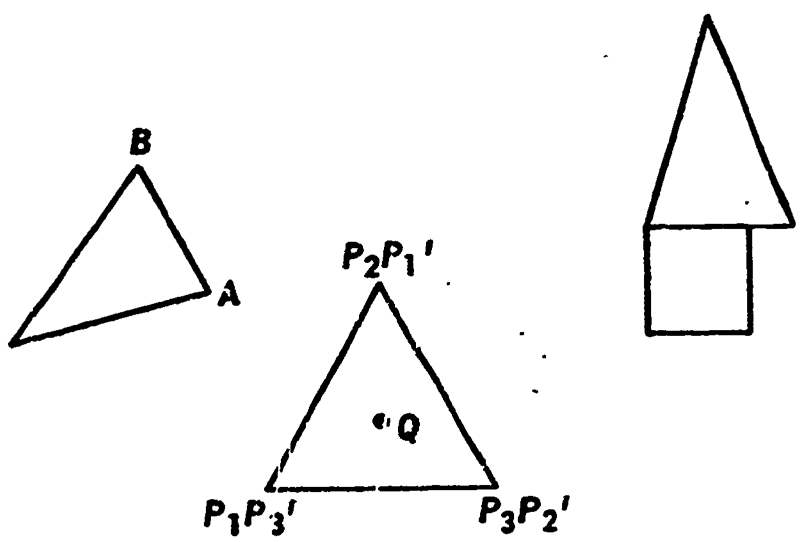

Fig. 19.

Here the new position of P_1 is the same as the old position of P_2, the new position of P_2 is the same as the old position of P_3, and the new position of P_3 is the same as the old position of P_1. Each point of the triangle has moved, but the whole triangle in its new position has come out exactly on top of its old position.

If, in a movement, a point (like the point Q in the example above) has its new position the same as its old position, we say that it is a *fixed-point* of that movement.

If, in a movement, a figure (like the triangle $P_1P_2P_3$ in the example above) has its new position fall exactly on its old position, we say that the figure is *invariant* under the movement.

PROBLEMS 6-2B

1. Consider the movement which carries $\triangle ABC$ to $\triangle A'B'C'$ in the following picture.

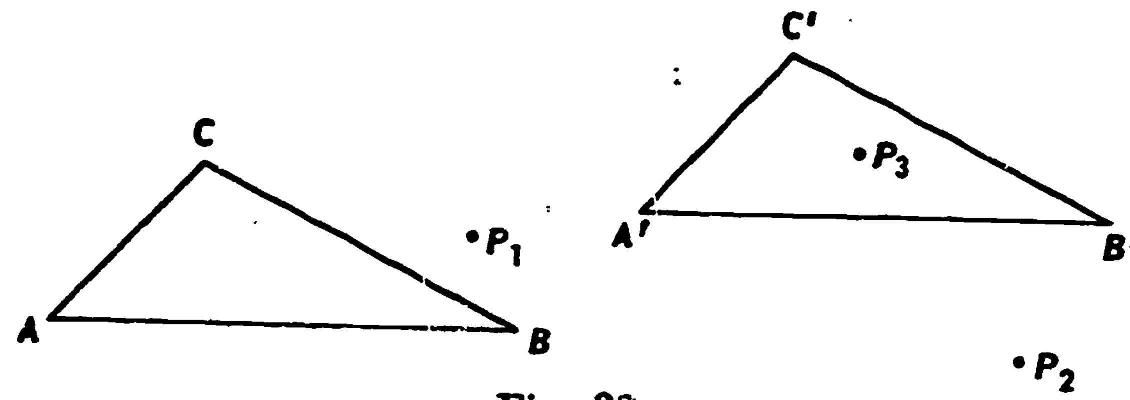

Fig. 20.

Where does it carry points P_1, P_2, and P_3? Does this movement have any fixed points?

2. Consider the movement which carries $\triangle ABC$ *to* $\triangle A'B'C'$.

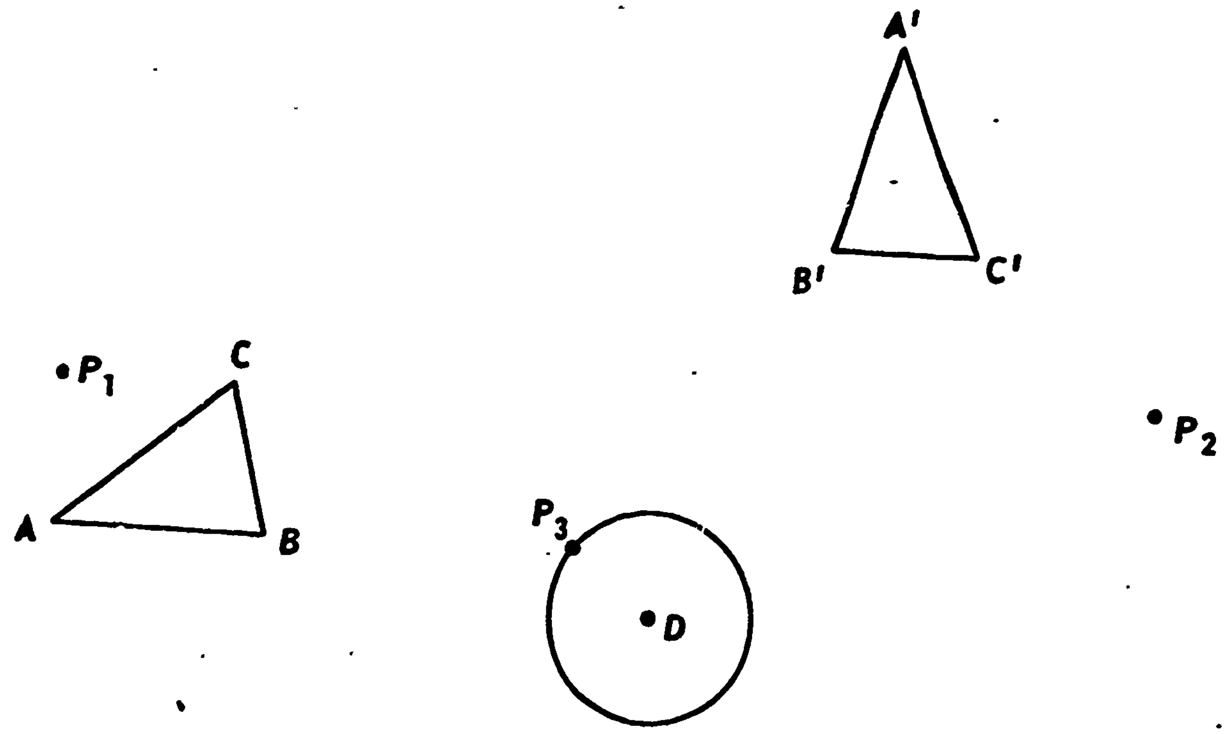

Fig. 21.

What does it do to the point P_1? To the point P_2? To the point P_3? To the circle D?

3. Consider the movement which carries ΔABC to $\Delta A'B'C'$ in the following picture.

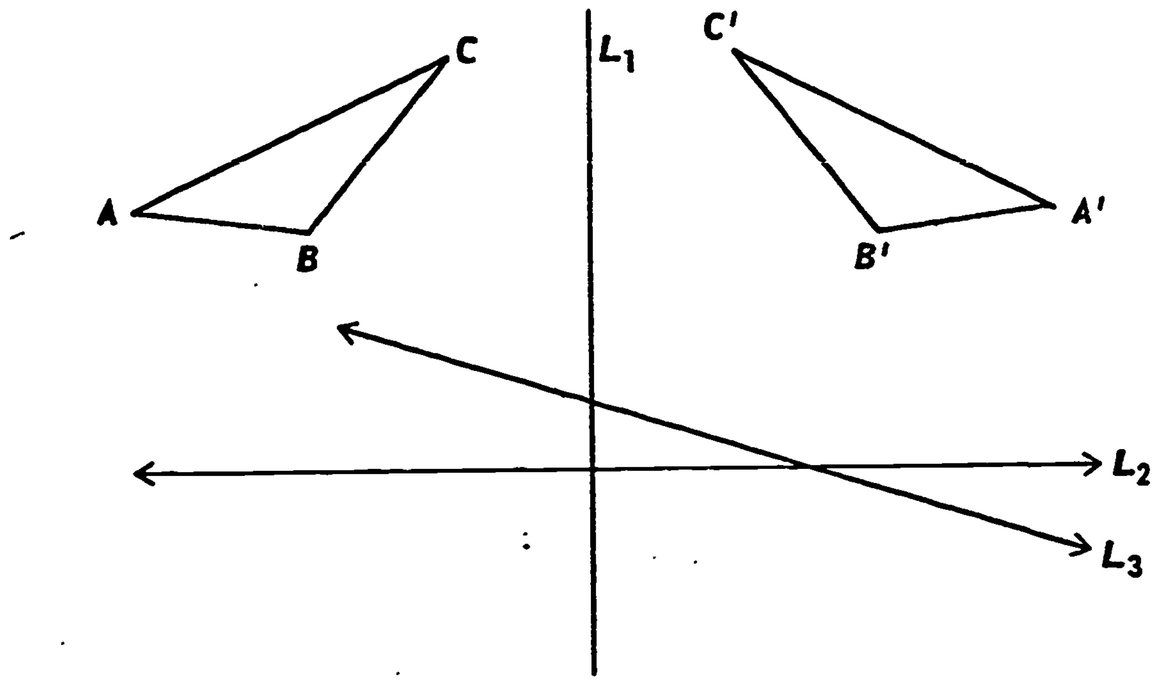

Fig. 22.

Where does it carry the line L_1? Where does it carry the line L_2? Where does it carry the line L_3? (Note: you will have to turn your tracing paper over to obtain this movement.)

Challenge Problem

4. a. In Problem 3, what are the fixed points of the movement?
   b. For the movement of Problem 3, describe all lines that are invariant under the movement.

5. How many distinct motions carry triangle A onto triangle B?

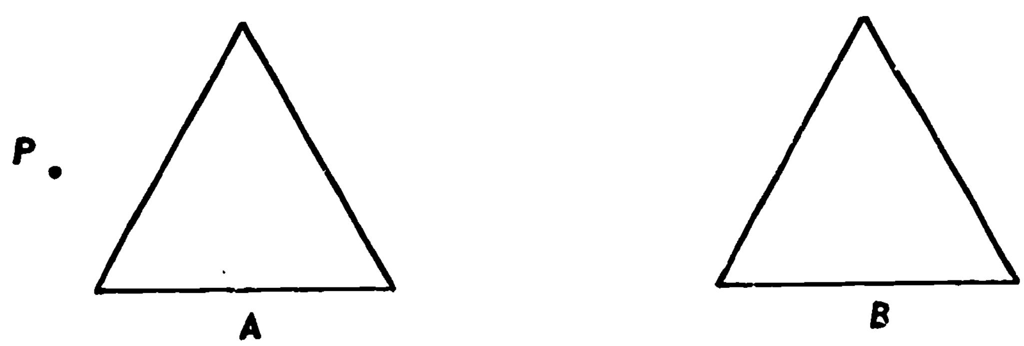

Fig. 23.

What does each of these movements do to the point P?

If we wish to move a figure like either of the two below,

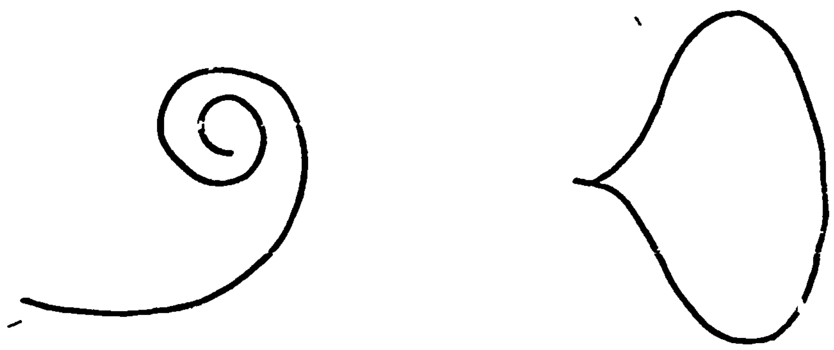

Fig. 24.

we must use tracing paper and prick through many points of the figure. But if we have a figure made up entirely of straight line segments and circular arcs, we can move it by ruler and compasses without using tracing paper at all. Our earlier work in geometry shows us how to do this. For example, in the following picture (which is the same as Figure 10 in this section)

Fig. 25.

we can move triangle ABC so that C falls on F and $\overline{AC}$ falls on the line of $\overline{EF}$ as follows. Extend segment $\overline{EF}$. Measure distance AC and mark off an equal distance from F on $\overline{EF}$ extended. This gives a point A' on $\overline{EF}$ extended. Draw an arc with centre F and radius BC. Draw an arc with centre A' and radius AB. Take the intersection of these arcs as B'. Draw the segments $\overline{A'B'}$ and $\overline{FB'}$. When you finish, your figure should look like this.

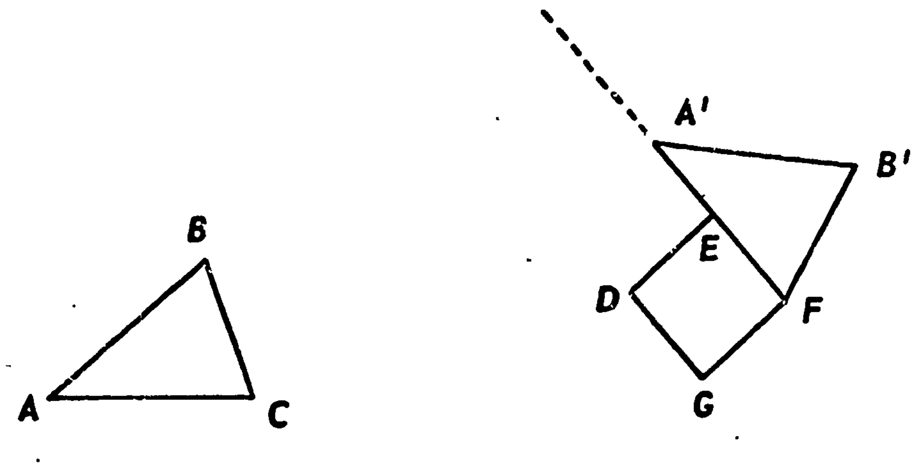

Fig. 26.

Here is another example. Move the line segment $\overline{AB}$ parallel to itself so that point A falls on point C in the following picture.

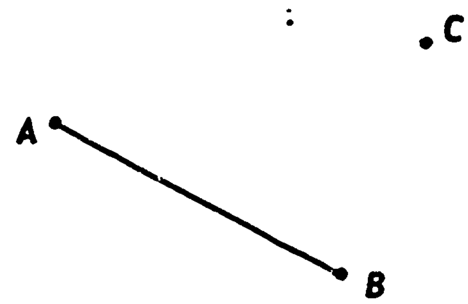

Fig. 27.

To solve this, draw segment $\overline{AC}$ and extend it beyond C to E. Then use compasses to obtain point D so that $\widehat{DCE}$ equals $\widehat{BAC}$ in measure. Draw $\overline{CD}$ and locate B' on its extension so that $CB' = AB$. When you are finished your figure will look like this.

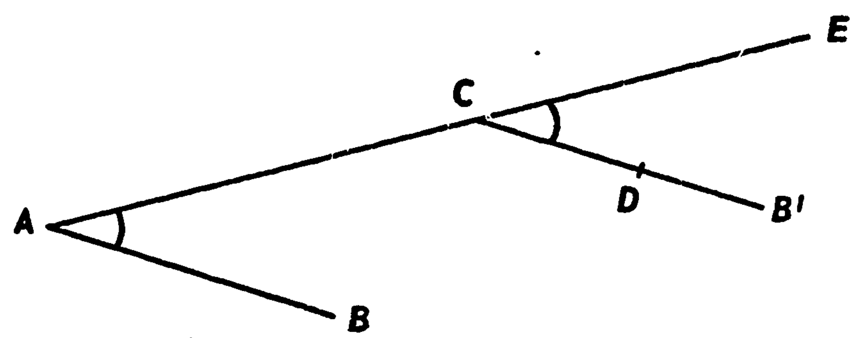

Fig. 28.

$\overline{CB'}$ is the new position of $\overline{AB}$ that we wanted.

Let ABC be a triangle. Assume that a certain movement of the tracing paper carries A to A', B to B', and C to C'. If we know the positions of ΔABC and $\Delta A'B'C'$, and if we are given any point P, we can then use compasses to find where the same movement carries P. The construction is very simple. (See Figures 29 and 30 below).

With B' as centre, draw a circle of radius BP. With A' as centre, draw a circle of radius AP. If P lies on $\overline{AB}$, then these circles intersect at one point, and we take this point of intersection for the new position of P.

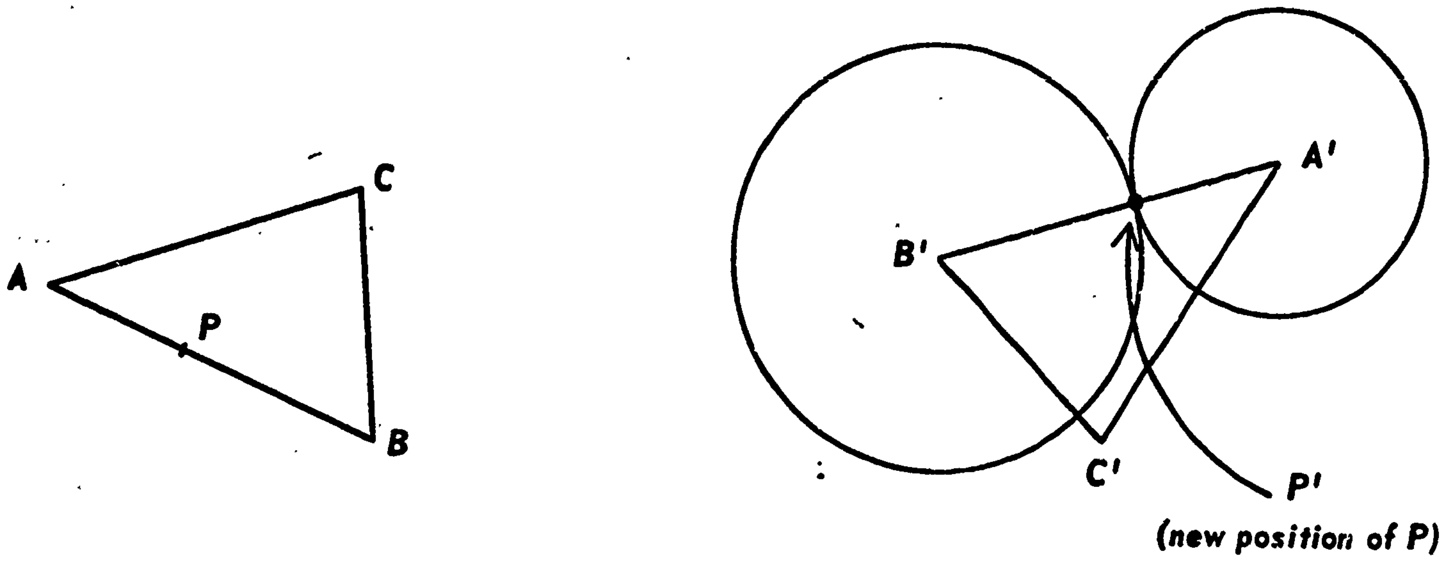

Fig. 29.

If P does not lie on $\overline{AB}$, then these circles intersect at two points which lie on opposite sides of $\overline{A'B'}$. In this case, see if P lies on the same side of $\overline{AB}$ as C. If so, choose, as the new position of P, that point of intersection which lies on the same side of $\overline{A'B'}$ as C'. If not, choose that point of intersection which lies on the opposite side of $\overline{A'B'}$ from C'.

The following picture shows this construction for the case where P is not on the same side of $\overline{AB}$ as C.

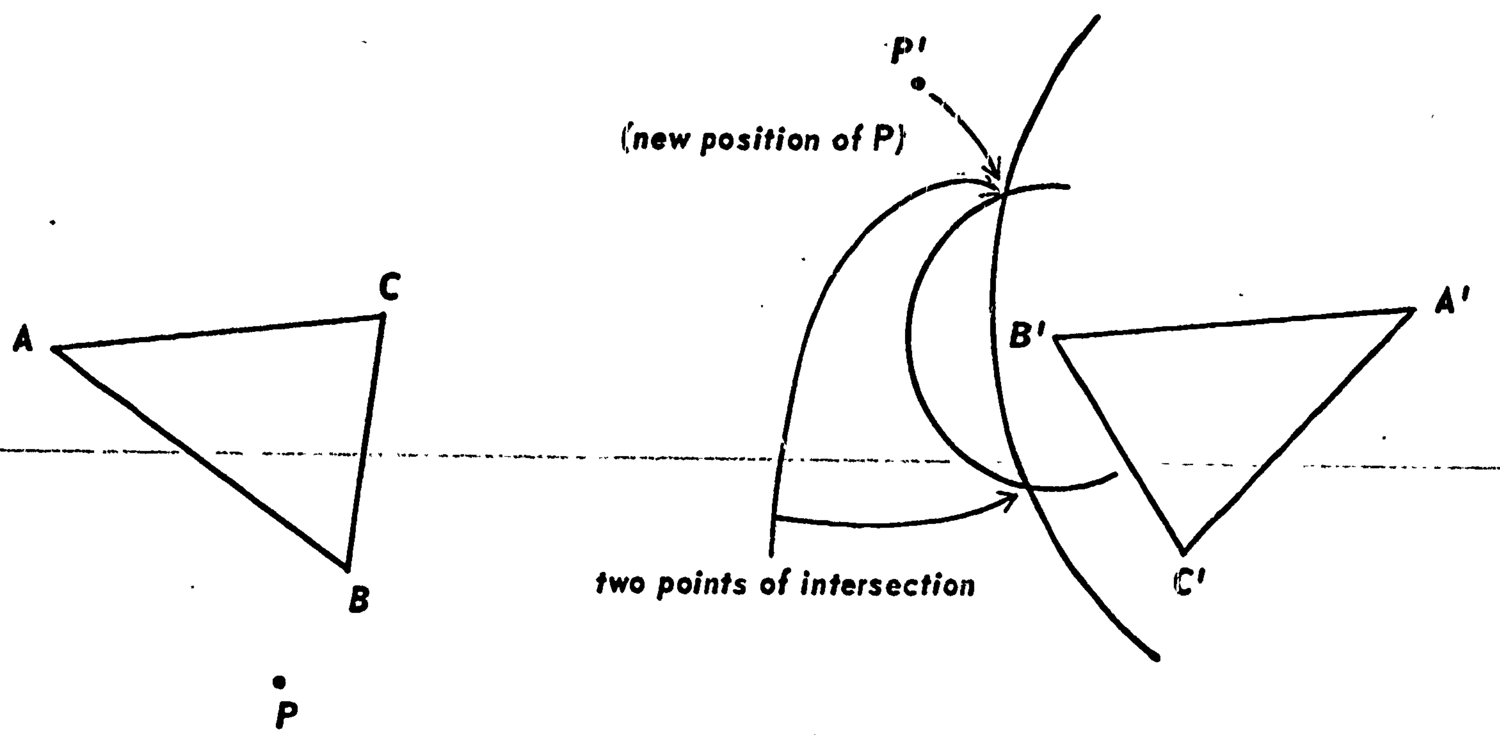

Fig. 30.

The use of ruler and compasses to move a figure or point often gives a more accurate result than the use of tracing paper. In the rest of this chapter we are going to study some of the things that happen when we move figures and points in the plane. We shall find that if we move a figure, or part of a figure, this can often help us to find new facts about the original figure. Often, in order to carry out a proof, it will be enough to imagine the movement and then make a rough sketch of the result, without using tracing paper or doing all the details of a ruler and compasses construction.

PROBLEMS 6-2C

1. Use ruler and compasses to do Problem 2 in Problems 6-2A.

2. Do the same for Problem 3 in Problems 6-2A.

3. Do the same for Problem 1 in Problems 6-2A.

4. In the following figure, a certain movement of the tracing paper carries ΔABC to $\Delta A'B'C'$. Use compasses to find where this movement carries the points D, E, and F.

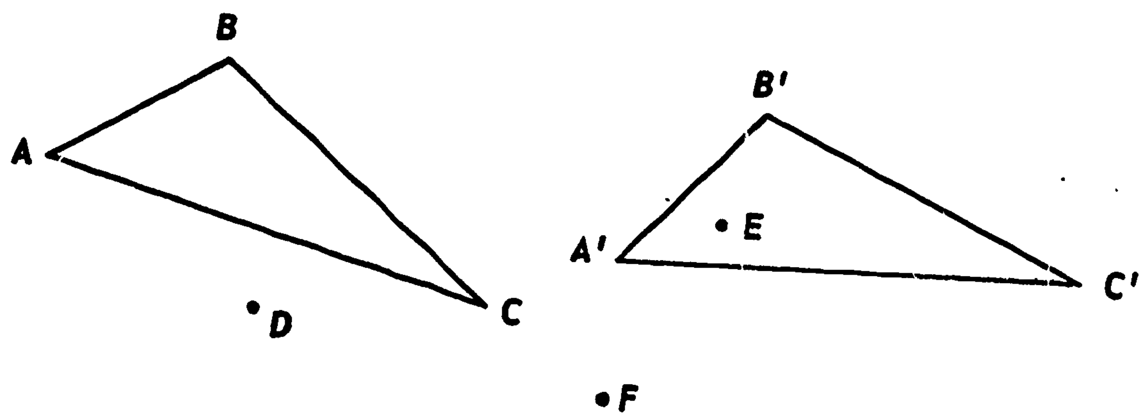

Fig. 31.

Where does this movement carry the triangle whose vertices are D, E, and F?

5. a. If a movement carries point A to point A' in the following picture, what are all the possible new positions of point P?

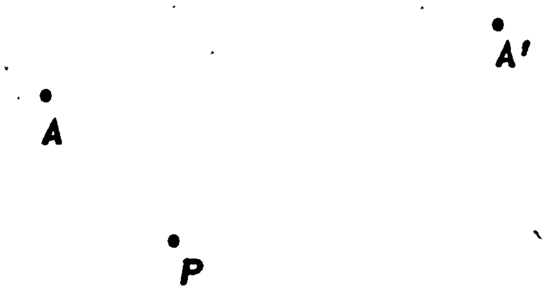

Fig. 32.

b. If a movement carries A to A' and B to B', what are all the possible new positions of the point P?

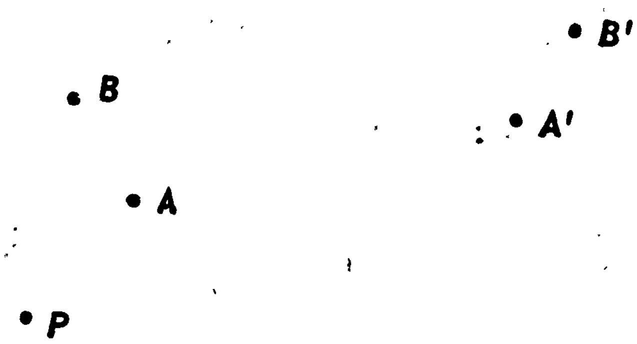

Fig. 33.

c. If a movement carries A to A', B to B', and C to C', what are all the possible new positions of the point P?

Fig. 34.

Challenge Problem

6. State a general rule which covers the cases described in Problem 5.

6-3 RIGID MOTIONS.

In the last section, we moved figures and points in the plane by sliding a piece of tracing paper to a new position. Let us now imagine a large (in fact infinite) piece of tracing paper on top of the plane. By copying from the plane onto the tracing paper and then sliding the tracing paper to a new position (or possibly turning it over and sliding it) we can move a figure in the plane. Indeed, given a change in position of the tracing paper, we can see where, in the plane, that change of position carries *any* given point, line, or figure that we are interested in. Such a change in position over the entire plane is called a *rigid motion*. Given a rigid motion and given a point P, the new position of P is called the *image* of the old position of P. Given a figure, the new position of the figure is called the *image* of the old position of that figure. If P' is the image of a point P under a given rigid motion, we sometimes say that the motion *maps* P to P'.

A rigid motion in geometry is much like a function in algebra. Just as a function on the real numbers is a way of assigning to every real number x a real number y called the *value* of the function at x, so a rigid motion T is a way of assigning to every point P in the plane a point P' called the *image* of P under the rigid motion. In algebra, if f is a function and x is a real number, we sometimes write the value of f at x as $f(x)$. Similarly, in geometry, if T is a rigid motion and P is a point, we sometimes write the image of P under T as $T(P)$. This notation gives us a short and useful way to say things about rigid motions. For example we can now restate the definition of *fixed-point* as follows.

P is a fixed-point of T if and only if $T(P) = P$.

We have spoken of a rigid motion as a change in position of an imaginary piece of tracing paper, and we have talked about *movement* of this paper. It would be better to speak of a rigid motion simply as a *new position* of the entire piece of tracing paper. In our study of geometry it does not matter what path we follow during the actual movement of the paper. All that matters is the final position of the paper. Thus we see that a rigid motion is, in fact, a function (of a certain kind) from the plane into itself.

What if we take our infinite tracing paper and do not move it at all? In this case, the new position of the infinite sheet of tracing paper is the same as the old position. Since there is no *change* in position, it is natural to ask whether or not we wish to call this function from the plane into the plane (where, for every point P, the image of P is P itself) a rigid motion. We shall find it useful to call this function a rigid motion, even though no change in position has occurred. This rigid motion is called the *identity motion*, and we often write it as I. Thus we have, in our notation:

$$\text{for any point } P, \quad I(P) = P.$$

(Hence every point is a fixed point of the identity motion.)

We now list several important facts about rigid motions. These facts hold for all rigid motions. Since they hold for all rigid motions, we say that they are *properties* of rigid motions. Let T be a rigid motion defined by the change in position of an infinite piece of tracing paper.

(1) If P and Q are points in the plane, if $P' = T(P)$, and if $Q' = T(Q)$, then the distance PQ is the same as the distance $P'Q'$.

(2) If P and Q are distinct points, if $P' = T(P)$, and if $Q' = T(Q)$, then P' is distinct from Q'.

(3) For every point Q there is a point P such that $Q = T(P)$. That is to say, every point is an image of some point.

(4) The image of any straight line is a straight line.

(5) If P, Q, and R are distinct points, and if $P' = T(P)$, $Q' = T(Q)$, and $R' = T(R)$, then $P'\widehat{Q'}R'$ is the image of $P\widehat{Q}R$ and $P'\widehat{Q'}R' = P\widehat{Q}R$ in measure.

(6) If two straight lines are parallel, then their images are parallel.

These six properties are obvious from the tracing paper idea of a rigid motion. Consider Property (6) for example. If two lines are parallel, then their copies on the tracing paper are parallel, and these copies remain parallel as we move the tracing paper. Hence the images of the lines must be parallel.

It is an interesting and surprising fact that, if we assume Property (1) for rigid motions (in other words, if we take Property (1) as a postulate), then we can prove Properties (2), (3), (4), (5), and (6) as theorems. We can do this *without using the idea of tracing paper at all*. These proofs will not be given here as part of your geometry course. If you are interested in looking at these proofs, you will find them in the appendix at the end of this chapter.

Since Property (1) is so basic, we give it a special name. It is called the *isometric property*. If a function from the plane into the plane has Property (1), we say that the function is *isometric*.

If a function is isometric, must there be a change in position of our infinite tracing paper which will give us the function? We shall see, in Section 6-6, that the answer to this question is *yes*. The idea of an *isometric function* and the idea of a *rigid motion* (as given by change in position of tracing paper) are therefore completely equivalent. Hence we could, if we wished, study rigid motions without using the idea of tracing paper at all. We would begin with the following formal definition.

> DEFINITION 6-1. A *rigid motion* is a function from the plane to the plane which is isometric.

We could then base our whole study on this formal definition. We will not do this here. In our work on rigid motions we will keep on using the tracing paper idea to help us to understand important facts and constructions.

A function from the plane into itself is often called a *mapping* on the plane. In geometry, this word "mapping" has exactly the same meaning as the word "function". As we have seen, if a mapping has the isometric property, it is called a *rigid motion*.

If a mapping has both Property (3) and Property (4) it is called a *transformation* on the plane. We have seen above that every rigid motion is a transformation. In Problem 6 below, we shall give an example of a transformation which is not a rigid motion.

In the rest of this chapter we shall study rigid motions in the plane. In later work in mathematics, we shall study other kinds of transformations as well. Transformations are important in both algebra and geometry.

In this chapter, we consider rigid motions of the plane onto itself. Rigid motions can also be used in space. In space, we can no longer use the tracing paper idea, but the formal definition of rigid motion can be given exactly as above. That is to say, rigid motion is a mapping from space into space with the property that for any points P_1 and P_2, if P_1' is the image of P_1 and P_2' is the image of P_2, then $P_1'P_2' = P_1P_2$. In what follows, our main study will be of rigid motions in the plane.

PROBLEMS 6-3

1. Let T be the rigid motion which takes $\Delta P_1P_2P_3$ to $\Delta P_1'P_2'P_3'$ in the following figure. Find the images of the points Q_1, Q_2, and Q_3 under T. What is the image of the square S under T?

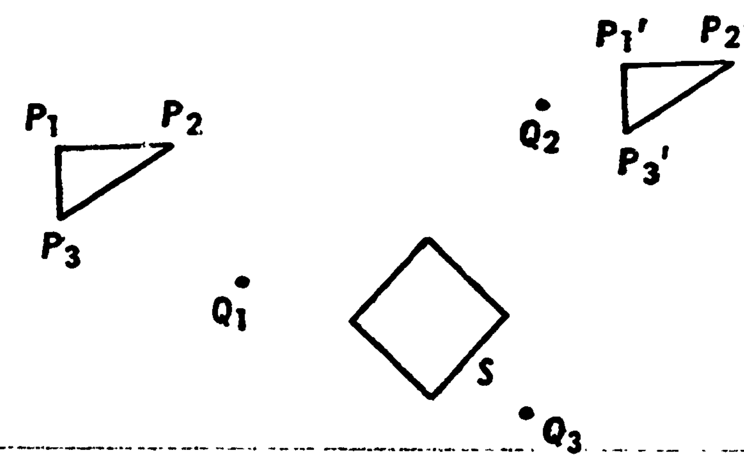

Fig. 35.

2. In the figure of Problem 1, what point has Q_1 as its image under T? Draw the triangle that has $\Delta P_1P_2P_3$ as its image under T.

3. Let T be the rigid motion which takes $\Delta P_1P_2P_3$ into $\Delta P_1'P_2'P_3'$ in the following figure.

Fig. 36.

What is the image of $\Delta P_1'P_2'P_3'$ under T? Can you draw the triangle which has $\Delta P_1P_2P_3$ as its image?

Challenge Problem

4. Let T be the rigid motion which takes $\Delta P_1P_2P_3$ into $\Delta P_1'P_2'P_3'$ in the following figure.

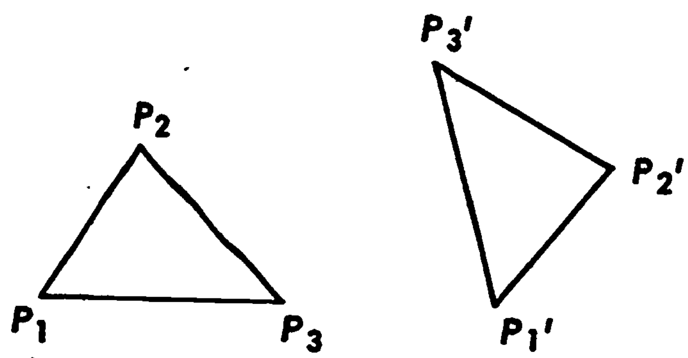

Fig. 37.

What is the image of $\Delta P_1'P_2'P_3'$ under T? Can you draw the triangle which has $\Delta P_1P_2P_3$ as its image?

5. Tell which of the following statements are true and which are false.
   (a) Every rigid motion has a fixed point.
   (b) There exists a rigid motion such that the image of each straight line is either identical with that line or parallel to it.
   (c) In every rigid motion there is some straight line which is parallel to its image.
   (d) In every rigid motion there is a line which is its own image.
   (e) There is a rigid motion which has no fixed point, but has a line which is its own image.
   (f) There is a rigid motion that has no fixed point, but which has a triangle that is its own image.

Challenge Problem

6. There are six different rigid motions which carry $\triangle ABC$ to $\triangle DEF$ in the following figure.

Fig. 38.

Locate the fixed points, if any, for each of these rigid motions.

7. Let L be a given line. We define a certain mapping T by showing, for any point P, how to find $T(P)$.
Let P be given. If P is on L, take $T(P)$ to be P itself. If P is not on L, drop a perpendicular from P to L, and locate a point on the perpendicular which is on the same side of L and twice as far from L as the point P. Take this new point to be $T(P)$.

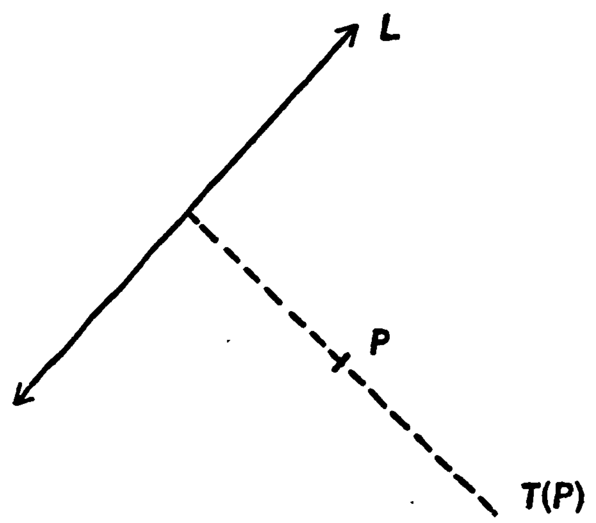

Fig. 39.

Show that this mapping is not a rigid motion. (Hint: show that this mapping is not isometric by finding two points P and Q such that

$PQ \neq P'Q'$, where $P' = T(P)$ and $Q' = T(Q)$. It then follows that T cannot be a rigid motion.)

NOTE: Tracing paper cannot be used to get this mapping. We can get it if, in place of tracing paper, we use a thin rubber sheet that can be stretched after we trace onto it.

Challenge Problem

8. Let L be a given line. We define a certain mapping T by showing, for any point P, how to find $T(P)$.

   Let P be given. If P is on L, take $T(P)$ to be P itself. If P is not on L, drop a perpendicular from P to L, extend the perpendicular beyond L, and locate a point on the perpendicular which is on the opposite side of L from P and at the same distance from L as P. Take this new point to be $T(P)$.

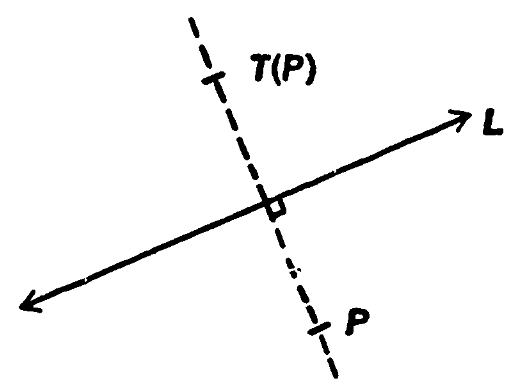

Fig. 40.

Show that this mapping is a rigid motion.

(Hint: show that the mapping is isometric by showing that for any two points P and Q, $PQ = P'Q'$, where $P' = T(P)$ and $Q' = T(Q)$. It then follows that T must be a rigid motion.)

How can we move our tracing paper to get the mapping T?

Challenge Problem

9. Let O be a given point in the plane. Then for every point P in the plane, we can locate a point P' (which will depend on P) as follows.

   If P is O, we take P' to be O. If P is not O, we draw a line from O through P and extend it beyond P. We then find a point P' on this line such that $OP = PP'$.

This procedure defines a mapping T such that for any P, $T(P) = P'$. Show that T is a transformation but not a rigid motion. (Hint: use similar triangles for proving Property (4).)

NOTE: As in Problem 7, tracing paper cannot be used to get this transformation, but we can get it if, in place of tracing paper, we use a thin rubber sheet that can be stretched after we trace onto it.

The transformation defined in Problem 9 also has Property (5). (Can you prove this?) If a transformation has Property (5), it is called a *motion*. Hence T, in Problem 9, is a motion which is not a rigid motion. Problem 7 gives an example of a transformation which is not a motion. See Problems 10 and 11.

10. Show that T in Problem 7 does not have Property (5).

Challenge Problem

11. Show that T in Problem 7 has Properties (3) and (4). (T is therefore a transformation.)

Challenge Problem

12. Prove that every transformation has Property (2) and Property (6).

6-4 USING MOTIONS TO SOLVE PROBLEMS.

There are many ways in which rigid motions can be used to help solve problems in geometry. We shall look at some of these in later sections. We give two examples here.

Example 1

The following figure shows a map of a river and a road. The figure also shows a scale of distance. It is desired to build a factory at a point on the river that is within two kilometers of the road. What points of the river can be used?

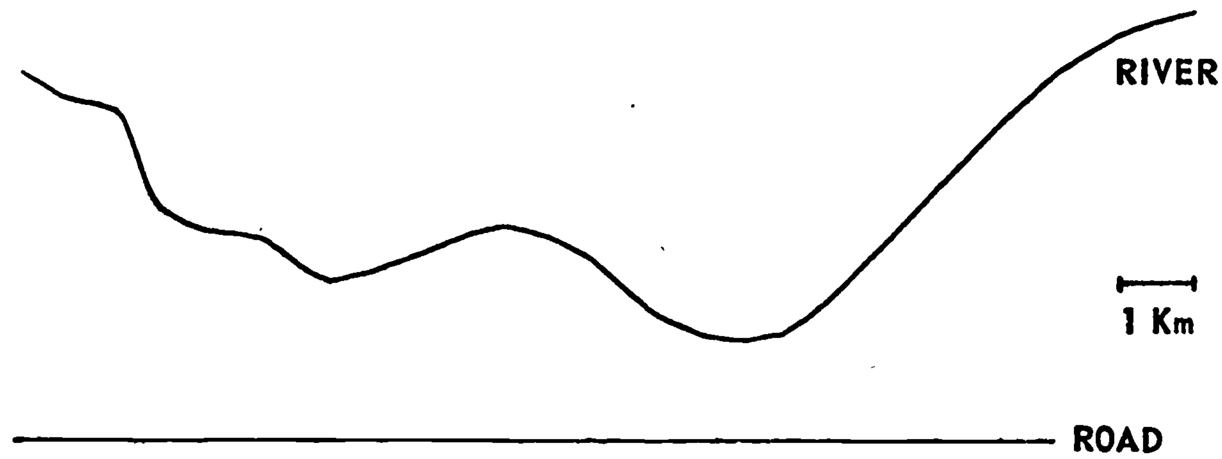

Fig. 41.

Solution to Example 1

The road is a straight line. Use the rigid motion which moves this straight line parallel to itself towards the river for a distance of two kilometres (on the scale). This gives the following figure.

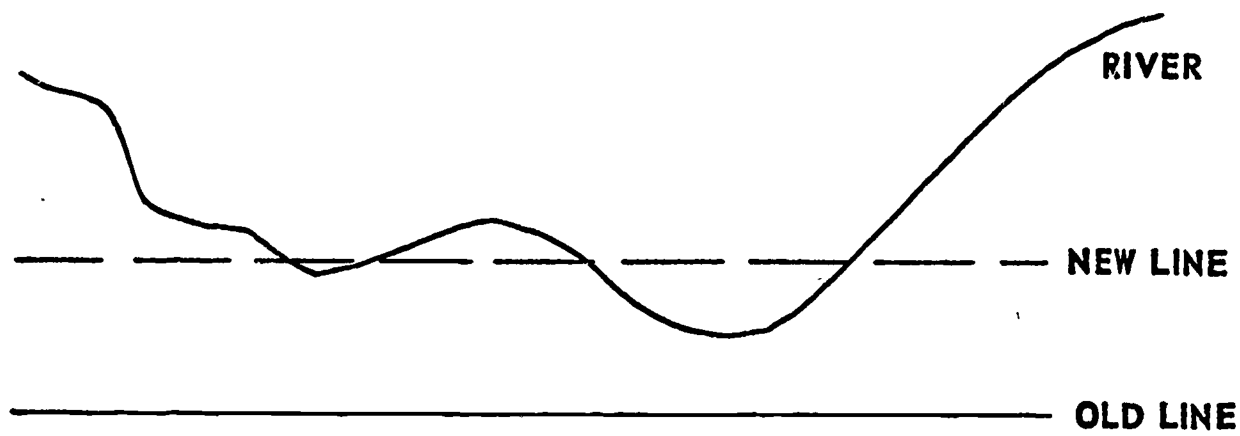

Fig. 42.

The portions of the river that lie below the image of the road now give the points that can be used for building the factory.

Example 2

In the following figure, construct a line segment which is parallel to line L, has length PQ, has one of its endpoints on circle C_1 and has one of its endpoints on circle C_2.

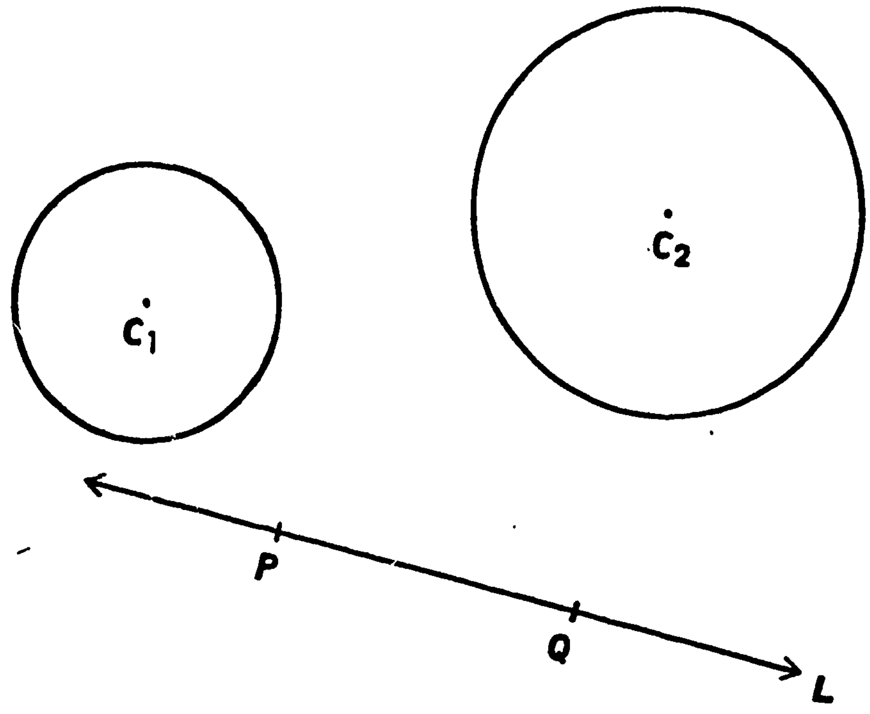

Fig. 43.

Solution to Example 2

Use a rigid motion which carries the circle C_1 a distance PQ towards C_2 in a direction parallel to L. The points of intersection of C_2 with the image of C_1 now give the possible points on C_2 for the segment to be constructed.

Challenge Problem

Use ruler and compasses to construct a line segment as asked for in the second example above. (There are two segments possible).

6-5 SPECIAL KINDS OF RIGID MOTION.

In the sections above, we have looked at different examples of rigid motions. Some of these rigid motions had special properties that others did not have. For example, some rigid motions had fixed points, others did not. It is helpful to have special names for some of the different kinds of rigid motions that are possible. In this section, we use the tracing paper idea to show the meaning of some of these names. In following sections we shall give more exact definitions.

(1) A rigid motion is called *direct* if we do not need to turn the tracing paper over in order to carry it out. If we do have to turn the tracing paper over, the rigid motion is called *reversing*.

(2) A rigid motion is called a *translation* if it is direct and can be obtained by sliding the tracing paper, without rotation, so that every point moves the same fixed distance.

(3) A rigid motion is called a *rotation* if it is direct and can be obtained by rotating the tracing paper about some chosen fixed point.

(4) A rigid motion is called a *reflection* if it is reversing and can be obtained by choosing a straight line and then turning the tracing paper over and putting it back down so that every point on the chosen line falls on its original position. The chosen line is called the *line of reflection* of the rigid motion.

Examples

(1) In Problem 1 of Problems 6-2A, the rigid motion which yields Figure 44 below

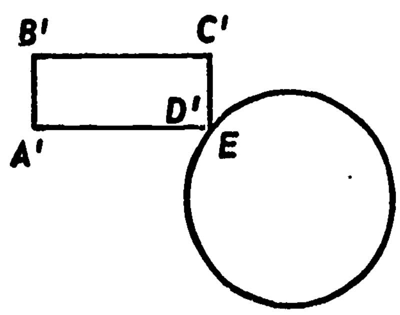

Fig. 44.

is a translation.

(2) In Problem 2, the rigid motion which yields Figure 45 below,

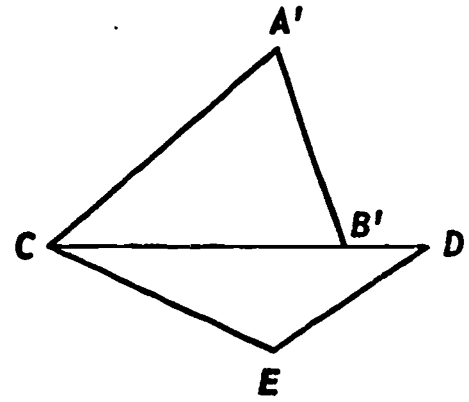

Fig. 45.

is a rotation. (Here C is the fixed point of the rotation.)

(3) In Problem 3, the transformation which yields Figure 46 below

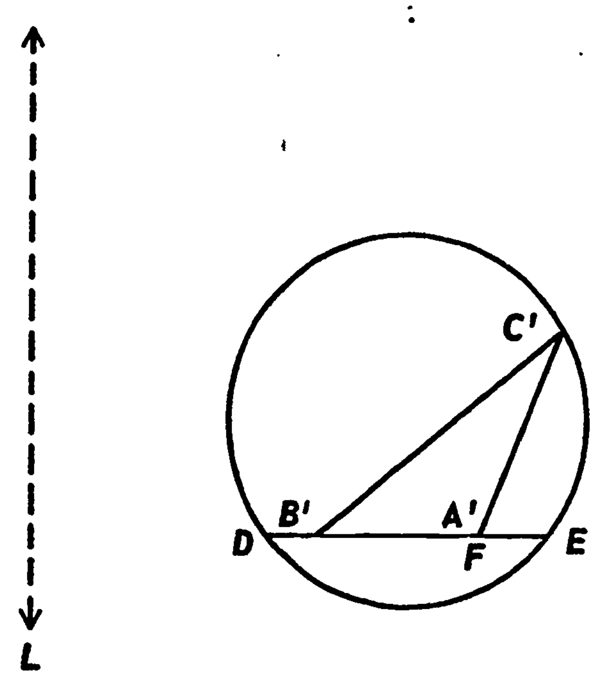

Fig. 46.

is a reflection. (Here L is the line of fixed points which we call the *line of reflection.*)

(4) In Problem 4 of Problems 6-3 we get an example of a rigid motion which is neither a translation nor a rotation nor a reflection. The rigid motion in this problem *can* be obtained by carrying out first a reflection and then a translation. In the following figure, the straight line indicates the line of reflection. Note that the translation leaves the line of reflection invariant.

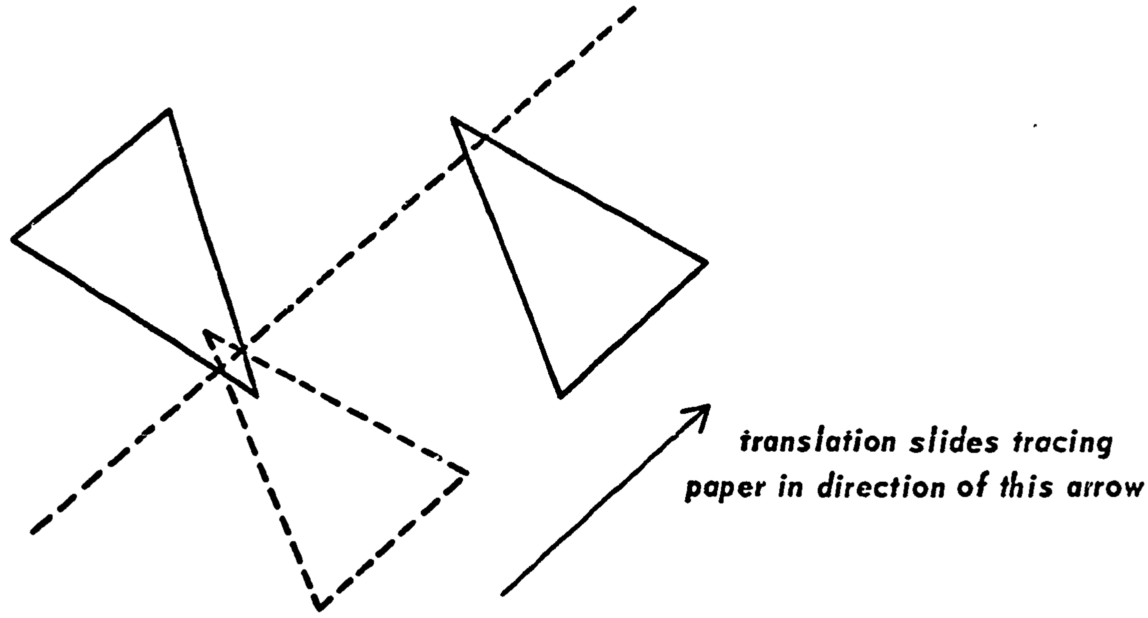

Fig. 47.

We shall see later that *every* rigid motion is either a translation, a rotation, a reflection, or can be obtained by carrying out first a reflection, and then a translation as in the above example.

Certain words and ideas are especially useful in talking about whether or not a motion is reversing. We give these words and ideas here.

Assume that we are looking down at the plane from above. Let A, B, C be the vertices of a triangle in the plane. We say that we see the vertices A, B, C in *clockwise order* if, as we look from A to B to C and back to A, our eyes follow the same direction that they would follow if we were watching hands move on the face of a clock. In the following picture, A, B, C occur in clockwise order.

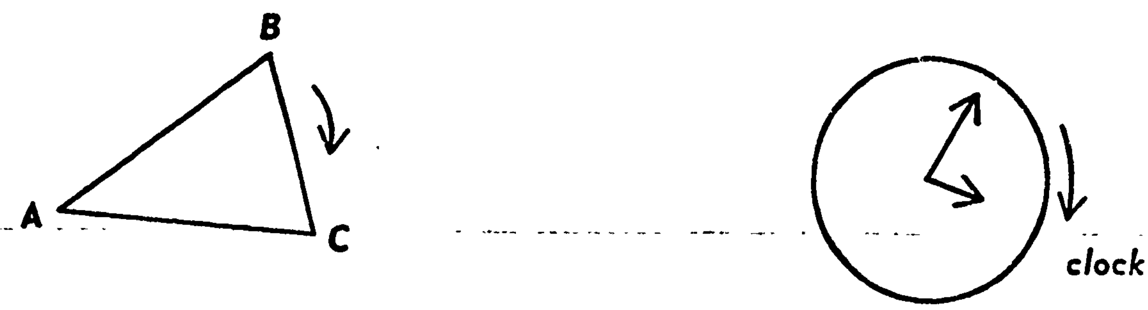

Fig. 48.

In the following picture, A, B, C do not occur in clockwise order.

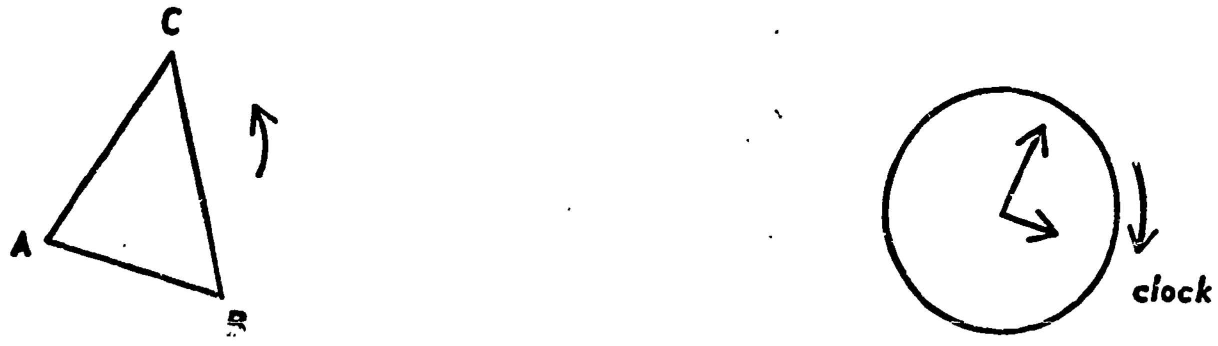

Fig. 49.

Whether or not vertices occur in clockwise order depends upon the order in which we name them. In the last picture above, A, C, B occur in clockwise order, while A, B, C do not.

If vertices do not occur in clockwise order, we say that they occur in *counter-clockwise order*. Thus, in the last picture above, we say that A, B, C occur in counter-clockwise order.

The following facts are now clear.

A rigid motion is direct if, for every triangle ABC in which A, B, C occur in clockwise order the images A', B', C' occur in clockwise order.

A rigid motion is reversing if, for every triangle ABC in which A, B, C occur in clockwise order, the images A', B', C' occur in counter-clockwise order.

PROBLEMS 6-5

1. Look at the rigid motions described in Problems 1, 2, 3 and 5 of Problems 6-2B. Which of these are direct and which are reversing? Which are translations, which are rotations and which are reflections?

2. We are given a rigid motion T. In the following figure, find $T(Q)$ assuming that T is a direct motion.

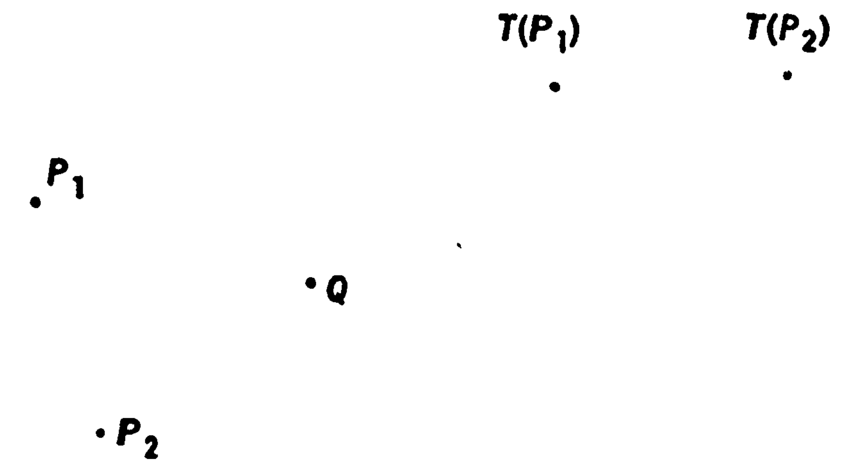

Fig. 50.

In the same figure, find $T(Q)$ assuming that T is a reversing motion.

3. T is a rigid motion; P_1, P_2 and Q are three distinct points.

   (a) You are given $T(P_1)$ and $T(P_2)$, and you are told that T is direct. Can you find $T(Q)$?

   (b) You are given $T(P_1)$ and $T(P_2)$, and you are told that T is reversing. Can you find $T(Q)$?

6-6 CONGRUENT FIGURES.

As we have seen, the image of a triangle under a rigid motion is congruent to the original triangle (by SSS). It is also true that for any two congruent triangles, there is a rigid motion which carries one triangle to the other. This fact is easy to see from the tracing paper idea of rigid motion.

Let ABC and $A'B'C'$ be two congruent triangles. We must show that there is a movement of tracing paper which carries $\triangle ABC$ to $\triangle A'B'C'$.

Imagine that we have made a tracing of $\triangle ABC$. Since the triangles are congruent, $AB = A'B'$. Slide the tracing paper so that A falls on A' and B falls on B'.

(i) If C (on the tracing) now falls on the same side of $\overline{A'B'}$ as C', we see that C must fall exactly on C' (since $C\widehat{A}B = C'\widehat{A'}B'$ in measure, and $C\widehat{B}A = C'\widehat{B'}A'$ in measure).

(ii) If, after the tracing paper is moved, C happens to fall on the opposite side of $\overline{A'B'}$ from C', turn the tracing paper over and place A over A' and B over B'. Now the new position of C falls on the same side of $\overline{A'B'}$ as C', and, as before, C must fall exactly on C'.

Thus, in either case, we have a movement of the tracing paper which carries ΔABC to $\Delta A'B'C'$. This movement defines the rigid motion that we wanted.

We thus have the following:

Two triangles are congruent if and only if there is a rigid motion which carries one triangle on to the other.

In the same way, we could show that two circles are congruent if and only if there is a rigid motion which carries one circle on to the other.

These facts suggest that we use the idea of rigid motion to *define* congruence for any two figures in the plane.

> *General definition of congruence.* Let S and S' be two sets of points in the plane. We say that S is *congruent* to S' if there is a rigid motion under which S' is the image of S.

This definition is different from the definitions of congruence for triangles and circles that you were given in your earlier work in geometry. We make the following two comments.

(a) For triangles and circles, the new definition agrees with the old definition. (That is to say, two triangles are congruent under the new definition if, and only if, they are congruent under the old definition.)

(b) The new definition is closer to the informal idea of "same size and shape" which led to our original study of congruent triangles. This informal idea was, simply, that two triangles are congruent if we can move one of them so that it falls exactly on the other.

NOTE ON ISOMETRIC MAPPINGS

In Section 6-3, we saw that every mapping obtained by using tracing paper must be isometric. We also stated that every isometric mapping can be obtained by using tracing paper. We can now see why this last fact is true.

Let T be a given isometric mapping. Then T has the property that it carries every straight line to a straight line. (This was Property (4) in Section 6-3. The fact that every isometric mapping has this property is proved in the Appendix to this chapter.) Let ABC be any given triangle. Let $A' = T(A)$, $B' = T(B)$, and $C' = T(C)$. Then the image of ΔABC under T must be $\Delta A'B'C'$, (by Property (4) of T); hence $\Delta A'B'C'$ is congruent to ΔABC (by the isometric property of T and SSS). What we did at the beginning of this section shows that there is a movement of tracing paper which carries A to A', B to B', and C to C'. Call the mapping over the plane given by this movement: T'. Then $T'(A) = T(A)$, $T'(B) = T(B)$, and $T'(C) = T(C)$. Now look at the construction given at the end of Section 6-2. This construction shows that if we know where an isometric mapping carries the vertices of a triangle, then we can find out exactly where that mapping carries any other point. Given any point P, the constructions for T and for T' must be the same. Hence, for any point P, $T(P) = T'(P)$. Hence T and T' are the same mapping. Thus our given mapping T can be obtained by a movement of tracing paper, which is what we set out to show.

6-7 TRANSLATIONS.

Let two points A and B be given. Draw an arrow from A to B.

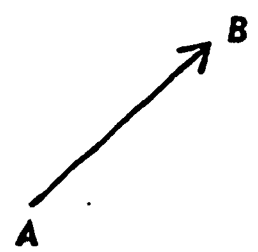

Fig. 51.

For any point P we carry out the following construction. Through P draw a line parallel to $\overleftrightarrow{AB}$. Call this line L'. (In case P lies on $\overleftrightarrow{AB}$, we take L' to be $\overleftrightarrow{AB}$ itself.) On L', we take a point P' so that $PP' = AB$ and P' lies in the same direction from P that B lies from A.

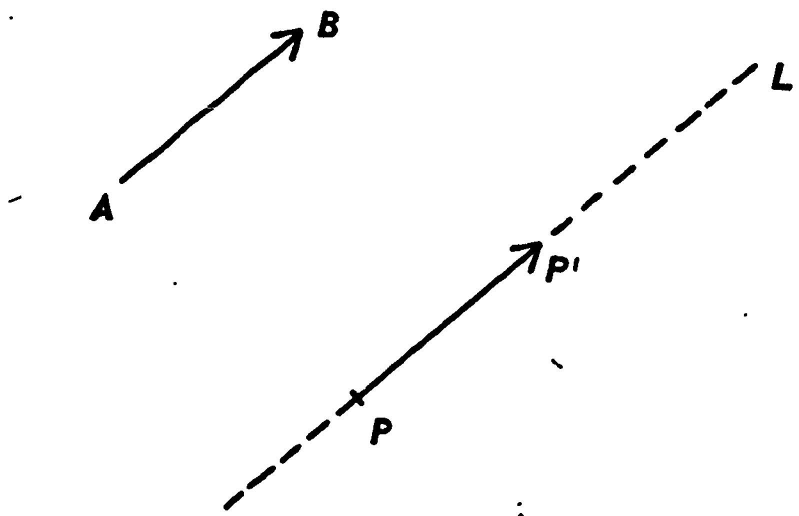

Fig. 52.

Note that in this construction, if we draw an arrow from P to P', we get an arrow with the same length and same parallel direction as the arrow from A to B.

An arrow from one point to another point in the plane is called a *vector*. We shall use the symbols $\mathbf{U}$ and $\mathbf{V}$ to stand for vectors. *NOTE*: Do not confuse a vector with a ray. A vector has *direction* and *length*, whereas a ray has direction only.

Let us give the name $\mathbf{U}$ to the vector from A to B in the above construction.

DEFINITION 6-2. If P' is obtained from P as in the above construction, we say that P' is the *result of translating* P *by the vector* $\mathbf{U}$.

Let a vector $\mathbf{U}$ be given in the plane. The construction above can be carried out for any point P. Hence we have a mapping T which we get by taking $T(P)$, for any point P, to be the result of translating P by the vector $\mathbf{U}$.

Must the mapping T, which we get in this way, be a rigid motion? If we can show that T is isometric, then, by our work in Sections 6-3 and 6-6, we will know that T is a rigid motion. We can show that T is isometric as follows. Take any points P_1 and P_2. The construction of P_1' and P_2' gives the following figure.

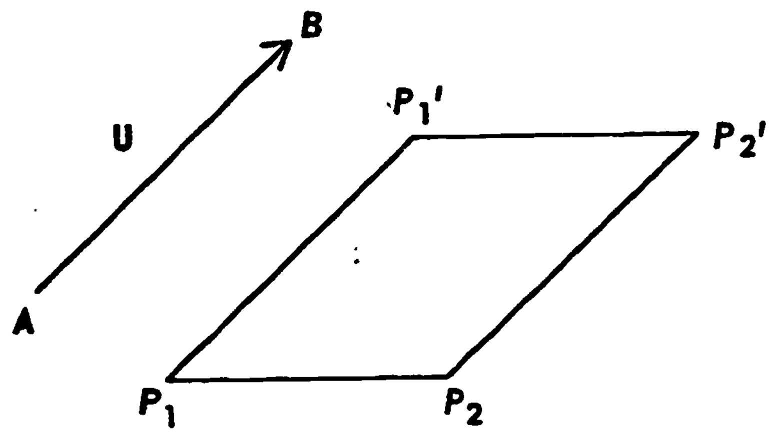

Fig. 53.

To show that T is isometric, we must show that $P_1P_2 = P_1'P_2'$.

Consider the quadrilateral $P_1P_1'P_2'P_2$. $\overline{P_1P_1'}$ is parallel to $\overline{P_2P_2'}$, since both are parallel to $\mathbf{U}$ by construction. $P_1P_1' = P_2P_2'$, since both are equal to the length of $\mathbf{U}$ by the construction. Hence $P_1P_1'P_2P_2'$ is a parallelogram. (Recall that any convex quadrilateral with a pair of opposite sides which are equal in length and parallel must be a parallelogram.) Hence $P_1P_2 = P_1'P_2'$, since opposite sides of a parallelogram must be of equal length. (In the figure, we have assumed that $\overleftrightarrow{P_1P_1'}$ and $\overleftrightarrow{P_2P_2'}$ are distinct lines. In case they are the same line, the proof is even simpler.)

We have shown T is isometric, and we see that T is a rigid motion. Is the rigid motion T direct or reversing? The following figure makes it clear that T must be direct.

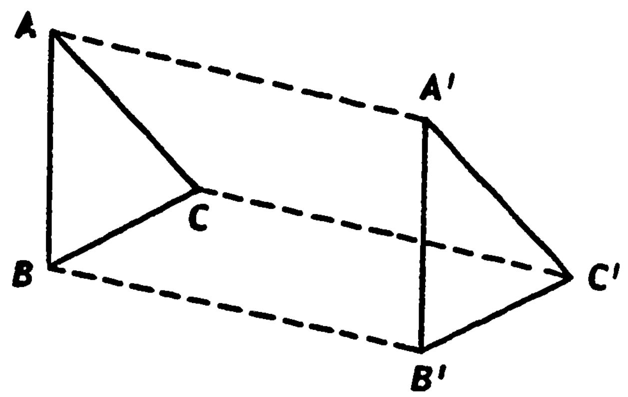

Fig. 54.

What kind of rigid motion is T? *Clearly, every such T is what we called, in Section 6-5, a translation, and every translation can be obtained by using a suitable vector* **U** *in the above construction for T.* The vector **U** simply gives the distance and direction through which we slide the tracing paper to get the rigid motion T.

We now list some more important facts about translations. In a translation, every point moves the same distance. This is clear from the construction.

Given any line L, the image of L in a translation is either parallel to L or else is L itself. To see that this is true, look at Figure 53 where we showed $P_1P_2 = P_1'P_2'$. If P_1 and P_2 are two points on L, then P_1' and P_2' are two points on the image of L, and, since $P_1P_1'P_2'P_2$ is a parallelogram, we have that L and its image are parallel. For what lines is it true that the image of L is L itself?

Is the identity motion a translation? We agree to call it a translation, although it is the translation in which every point moves zero distance. It is given by a vector whose length is zero.

TRANSLATIONS AND COORDINATE AXES

Suppose we take a point O which is the origin of a coordinate system. Then we can draw our coordinate axes as in Figure 55.

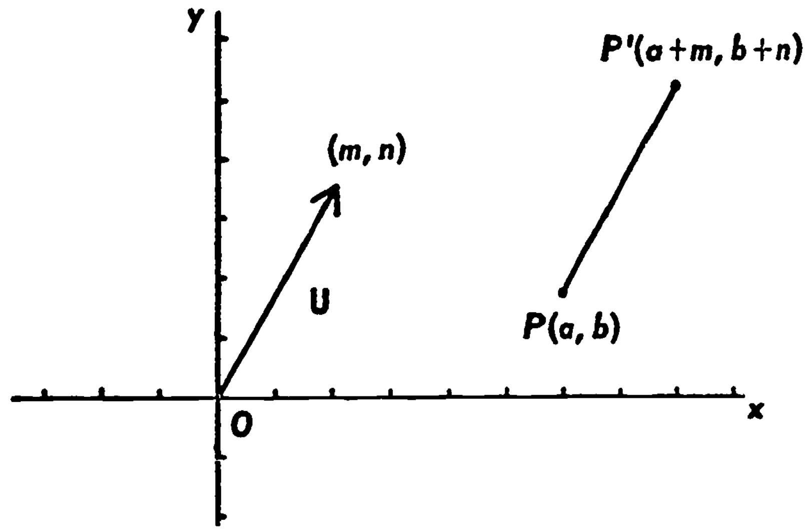

Fig. 55.

Suppose we are given a vector **U** from the point O to some point with coordinates (m, n). Let P have coordinates (a, b). Let P' be the result of translating P by **U**. What are the coordinates of P'? Clearly P' has coordinates $(a + m,\ b + n)$. Can you show this?

PROBLEMS 6-7

1. Let T be the translation given by the vector **U** in the following figure. Find the image of the triangle. Find the image of the circle. Find the figure which has the circle as its image.

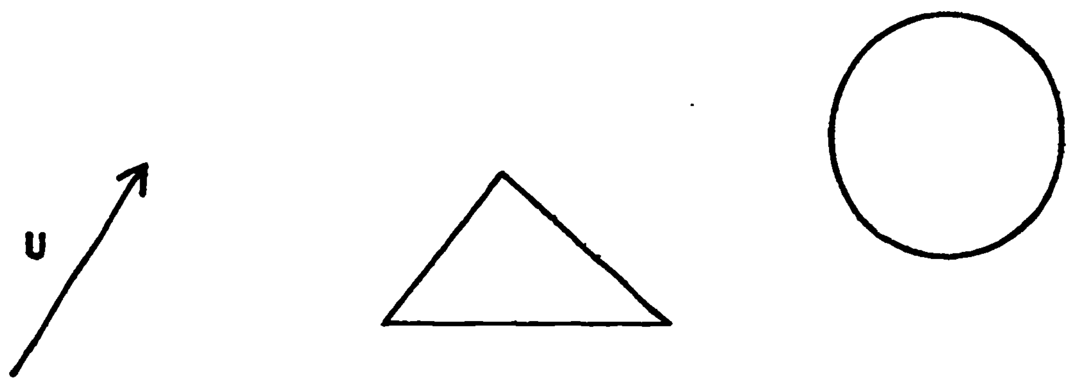

Fig. 56.

2. In Problem 1, what lines are carried into themselves by T (are invariant under T)?

3. What is the result of applying first translation **U** and then translation **V** in the following figure?

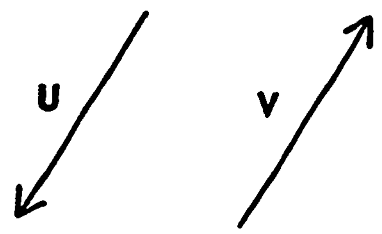

Fig. 57.

4. What is the result of applying first translation **U** and then translation **V** in the following figure?

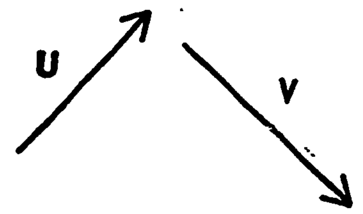

Fig. 58.

5. Let **U** be the vector between (0, 0) and (2, -4). For each of the following points P, find the coordinates of P' where P' is the result of translating P by **U**.

   P. (1, 1)
   (-2, 4)
   (7, -10)

6. In the last paragraph of this section (Figure 55), prove that P' has coordinates $(a + m,\ b + n)$.

Challenge Problem

7. You are told that a certain rigid motion T has the property that it carries every point the same distance. Show that T must be a translation.

8. You are told that a certain rigid motion T has the property that, for every line L, the image of L is either parallel to L, or is L itself. Show that T need not be a translation.

9. You are given two points P_1 and P_2, you are told that a certain rigid motion T carries P_1 to P_1' and P_2 to P_2', and you are told that $P_1P_1'P_2'P_2$ is a parallelogram. Show that T need not be a translation.

Challenge Problem

10. You are given the same information as in Problem 9 and you are also told that T is direct. Show that T must be a translation.

6-8 ROTATIONS.

Suppose we have a fixed point O and a point P. If we move P to the position P' (as in Figure 59) in such a way that $OP = OP'$ and $m(\widehat{POP'}) = \beta$, then we say that we have turned P through an angle β about O.

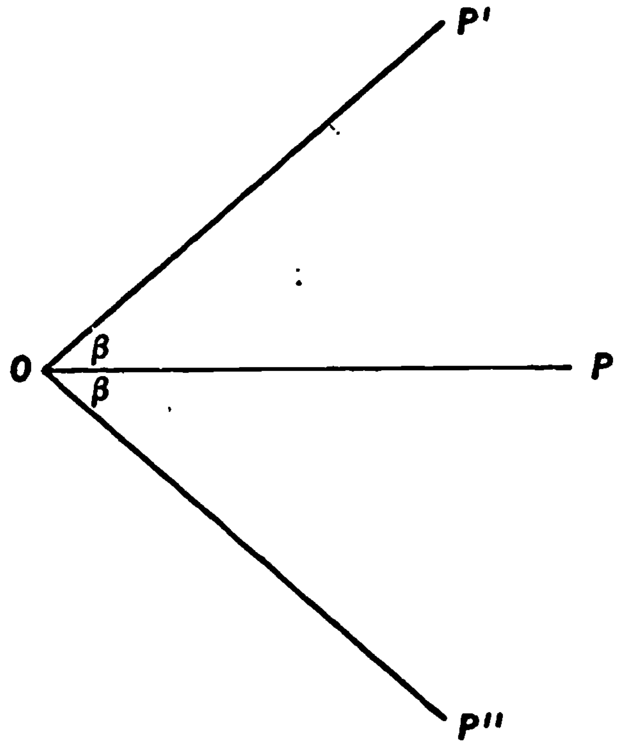

Fig. 59.

We can also turn P through an angle β to get P''. So it is possible to turn P through the same angle β, but getting two different positions as P' and P'' in Figure 59.

In order to get just one result when we talk about rotating a point through a given angle β about a fixed point, we need to give also the direction in which the angle β is turned.

For example in the above case when P moves to P', we actually turned in the direction $\circlearrowleft$ that is the counter-clockwise direction. To get P'' we would turn in the direction $\circlearrowright$. We see clearly that $\circlearrowright$ is in the opposite direction to $\circlearrowleft$. So we use the following sign convention: the direction $\circlearrowleft$ (counter-clockwise) will be taken as *positive* and the opposite direction $\circlearrowright$ (clockwise) will be taken as *negative*.

Whenever we say that a point (or for that matter anything else) is turned through an angle $\beta > 0$, we mean that the turning is in the counter-clockwise direction, and when we say that a point is turned through an angle $\beta < 0$, we mean that the turning is in a clockwise direction.

> DEFINITION 6-3. The point P is *rotated about O through an angle β*, if P moves in such a way that it remains a fixed distance from O while $\overrightarrow{OP}$ turns through the angle β.

Let a point O be given in the plane and let a value of β be given. For any point P we can find a new point P' by rotating P about O through an angle β. Hence we have a mapping T which we get by taking $T(P)$, for any point P, to be the result of rotating P about O through an angle β.

Must the mapping T, which we get in this way, be a rigid motion? If we can show that T is isometric, then, by our work in Sections 6-3 and 6-6, we will know that T is a rigid motion. We can show that T is isometric as follows. Take any points P_1 and P_2. The construction of P_1' and P_2' gives the following figure.

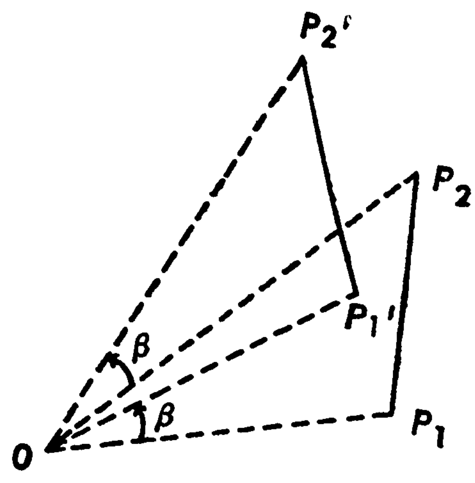

Fig. 60.

To show that T is isometric, we must show that $P_1P_2 = P_1'P_2'$.

In this figure, $m(P_1\widehat{O}P_2) = \beta + m(P_1'\widehat{O}P_2)$ and $m(P_1'\widehat{O}P_2') = \beta + m(P_1'\widehat{O}P_2)$. Hence $m(P_1\widehat{O}P_2) = m(P_1'\widehat{O}P_2')$. Also, $OP_2 = OP_2'$ and $OP_1 = OP_1'$, by the construction. Hence, by SAS, $\triangle P_1'OP_2'$ is congruent to

ΔP_1OP_2. It follows that $P_1P_2 = P_1'P_2'$. (In the figure, we have assumed that the interiors of $P_1\hat{O}P_2$ and $P_1'\hat{O}P_2'$ overlap. In case they do not overlap, the proof is the same except that $m(P_1\hat{O}P_2) = m(P_1'\hat{O}P_2') = \beta - m(P_1'\hat{O}P_2)$.)

Therefore, T is isometric, and we see that T is a rigid motion. Is the rigid motion T direct or reversing? The following figure makes it clear that T must be direct.

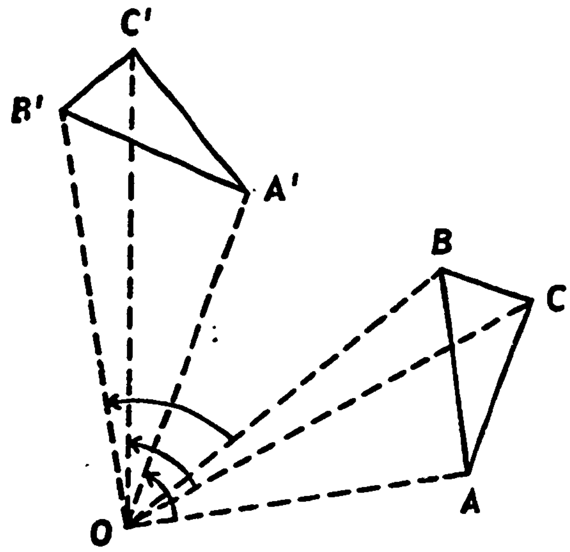

Fig. 61.

What kind of rigid motion is T? Clearly every such T is what we called, in Section 6-5, a rotation, and every rotation can be obtained by using a suitable point O and value β in the above construction of T. The point O is simply the point about which we rotate the tracing paper (imagine a pin stuck through the tracing paper at this point), and β is the measure of the angle (positive or negative) through which we turn the tracing paper.

If $\beta = 90^\circ$ in a rotation, we call that rotation a *quarter turn*. If $\beta = 180^\circ$ in a rotation, we call that rotation a *half turn*. Some of the problems below have to do with these special rotations.

The identity motion is the rotation that we get for $\beta = 0^\circ$. (The identity motion is thus called *both* a rotation and a translation.)

Note the following fundamental fact about rotations. Let T be a rotation through an angle β. Let L be any line. Let L' be the image of L under T. Then L and L' form an angle whose measure is β.

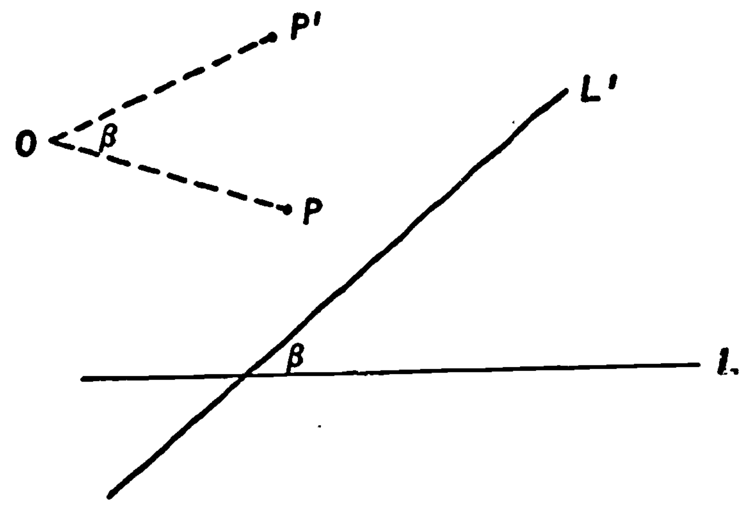

Fig. 62.

PROBLEMS 6-8A

1. If A is rotated 90^0 about O, where will its image A' be?

Fig. 63.

2. If B is rotated 180^0 about X, where will its image B' be?

Fig. 64.

3. If $\overline{AB}$ is rotated 90^0 about O where will $\overline{A'B'}$ be?

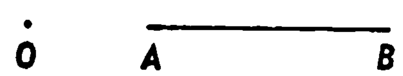

Fig. 65.

4. Where will a half-turn about O bring $\overline{AB}$ in Figure 66? If $\overline{A'B'}$ is the image of $\overline{AB}$ from this rotation and you moved towards the right to get from A to B, in what direction would you move to get from A' to B'?

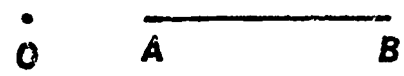

Fig. 66.

5. Where will a quarter turn about O bring $\overline{AB}$?

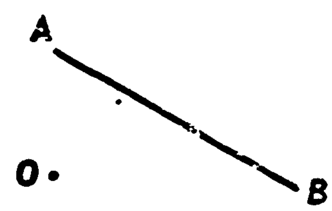

Fig. 67.

6. Draw $\overline{A'B'}$, the segment resulting from a half turn of $\overline{AB}$ about O. Can you show that $ABA'B'$ is a parallelogram?

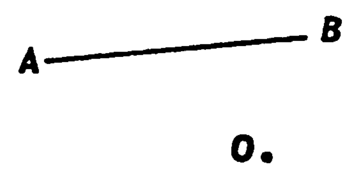

Fig. 68.

7. Let T be a rotation of 120^0 about O, the centre of the equilateral triangle. Find the images under T of the circle, and the two triangles in the following figure.

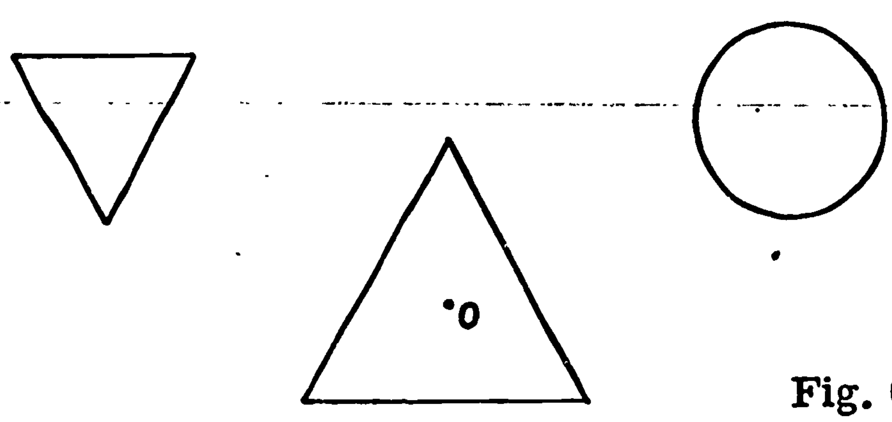

Fig. 69.

8. If, in the figure, A' is the image of A under a rotation about a point O which lies along $\overline{AA'}$, where is this point O?

Fig. 70.

9. Let A and A' be given. If A is moved to A' by a rotation about some point O, what is the locus of possible points O?

Challenge Problem

10. If $\overline{AB}$ is transformed into $\overline{A'B'}$ by a rotation about a point O, how can you find the position of O? Is there a rigid motion which carries $\overline{AB}$ to $\overline{A'B'}$, but is not a rotation about some point O?

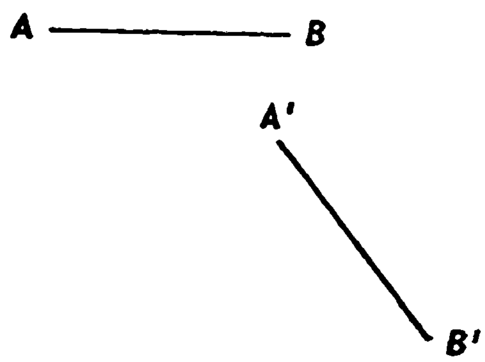

Fig. 71.

11. Triangle $A'B'C'$ is the image of ΔABC under a rotation. How would you find the point about which it was rotated?

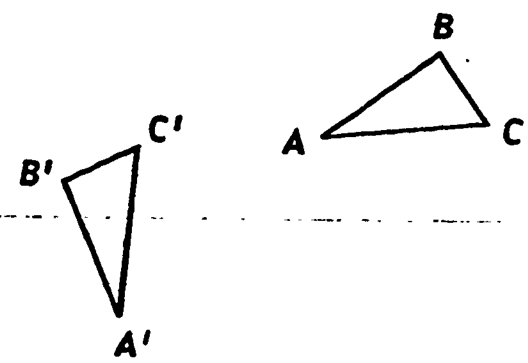

Fig. 72.

NOTE: The point about which a rotation is made is known as a centre of rotation.

12. Prove that in a rotation through angle β any line makes an angle β with its image.

ROTATIONAL SYMMETRY

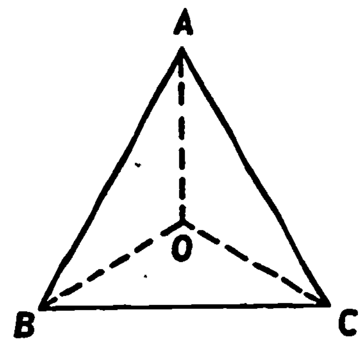

Fig. 73.

Consider the equilateral triangle ABC, with centre O. Suppose we rotate the triangle through 120^0 about O. Where will the sides of the image, $\Delta A'B'C'$, lie?

You will find they lie on the sides of ΔABC, although now C' will be where A was, A' will be where B was and B' will be where C was. That is, this image will fall exactly on the original triangle. Also if we rotate the triangle through 240^0, we find that the sides of the image again fall exactly on the original triangle.

Suppose we next take a square $ABCD$ with centre O. If we rotate the square 120^0 about O, will the sides of the image fall on the sides of the original square? What happens when the square makes a quarter-turn about O?

Consider the following figure. Could you find a rotation about O which will give an image lying on top of the original figure? (Find as many angles as possible lying between O and 360^0 for which this could be done.)

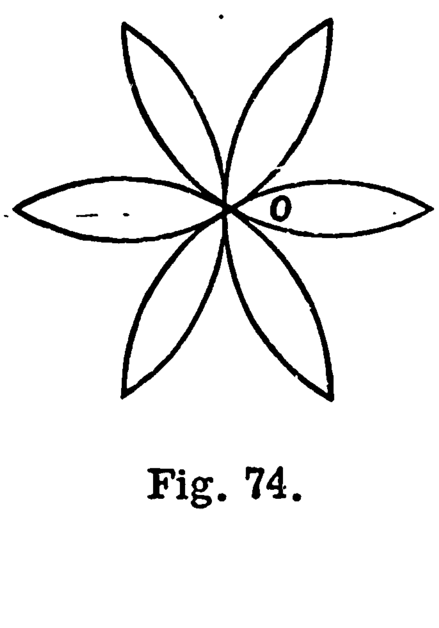

Fig. 74.

If we restrict ourselves to a rotation less than a complete turn (that is, less than 360^0) do you think you can find a rotation about O which will give an image coincident with the original figure in this illustration?

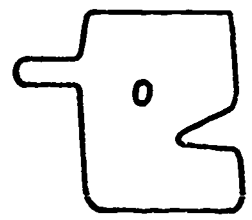

Fig. 75.

When we do a complete turn, every point returns to itself (and so we have the identity motion). So a complete turn will always leave a point, a line or any figure unchanged. If it is possible to rotate a figure less than 360^0 about a point O and bring it into coincidence with itself, then we say that the figure has *rotational symmetry* about O.

ROTATIONS AND COORDINATE AXES

Suppose we consider a point O which is the origin of a coordinate system, and we draw the coordinate axes as in the figure. A point R on the x-axis, when rotated 90^0 about O, goes to R'. What is the distance of R' from O? Suppose we considered another point P distant p from O along the y-axis. Where will its image P' lie under a quarter-turn about O? Where will its image P'' lie under a half-turn about O?

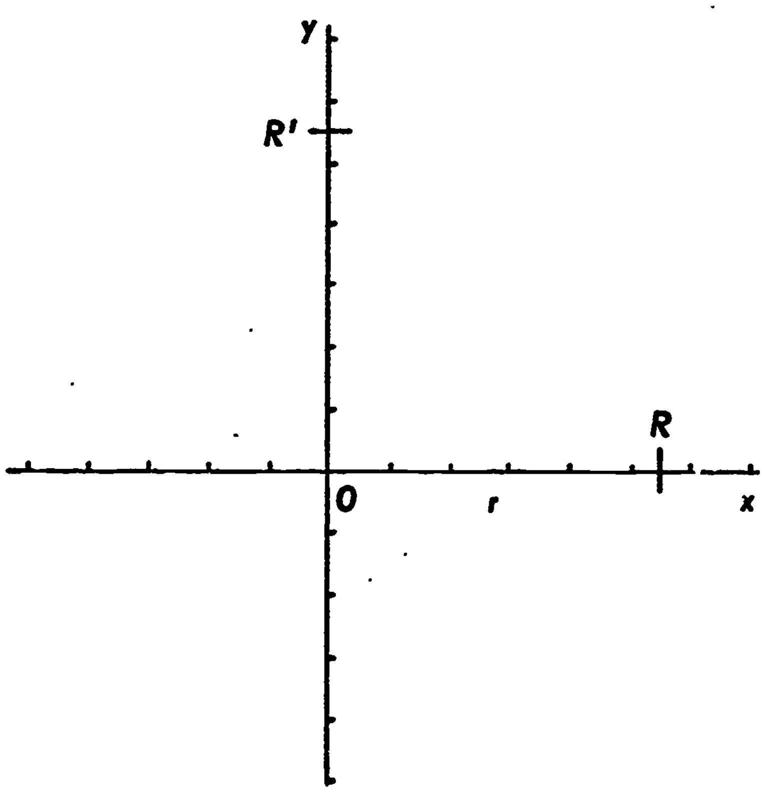

Fig. 76.

We can see that a half-turn will take R to R'', where R is a distance r from O along the positive x-axis and R'' is a distance r from O along the negative x-axis. A quarter turn will move R to R', where R' is a distance r from O along the positive y-axis. Can you make similar statements for the point P?

Let a point P have coordinates (a, b). What do the coordinates of P become under a half turn about the origin? Under a quarter-turn about the origin?

PROBLEMS 6-8B

1. Which of the following figures have rotational symmetry?

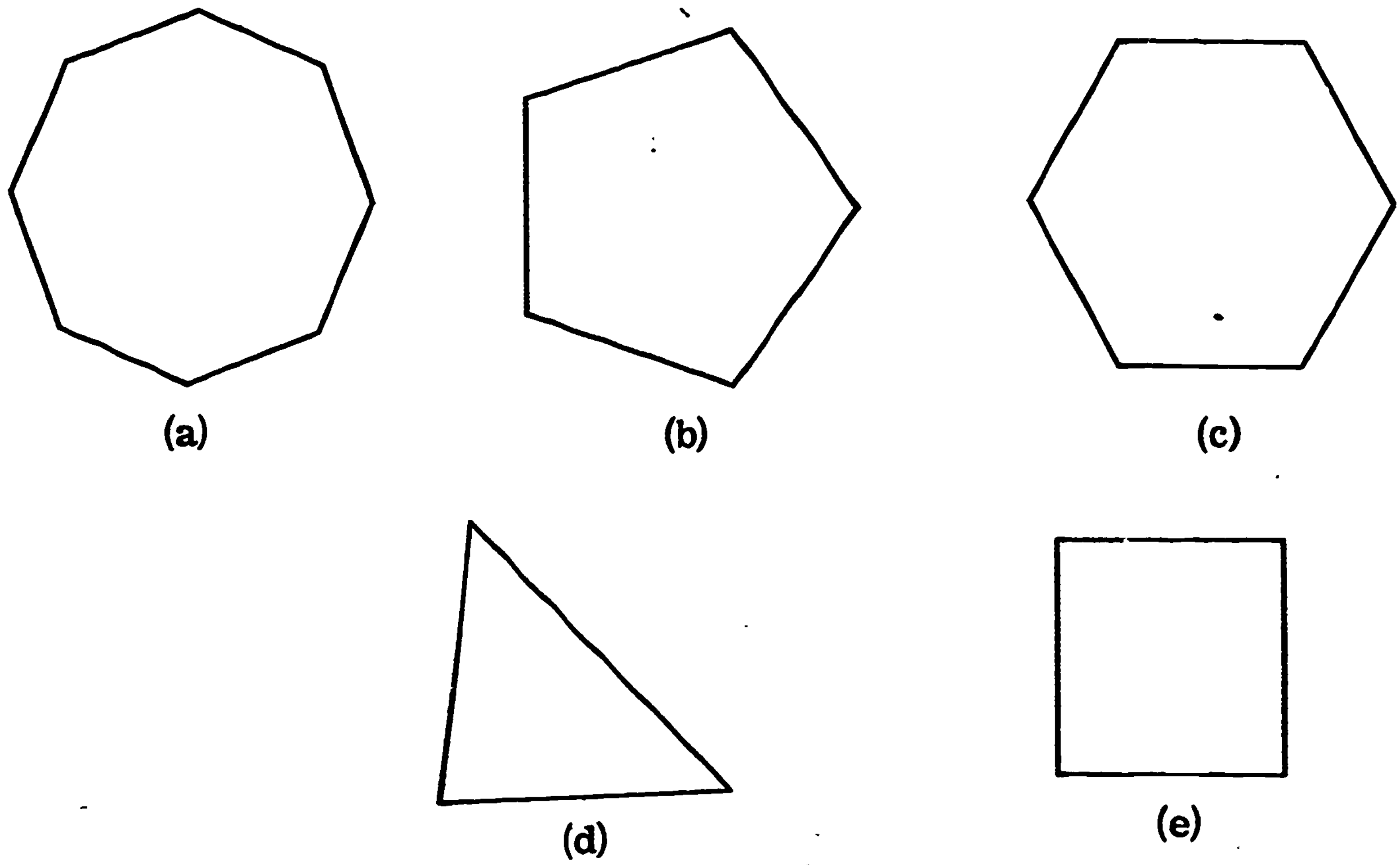

Fig. 77.

2. The number of distinct rotations that carry a figure onto itself for rotations less than 360^0 is called the *order of symmetry* of that figure. Find the order of symmetry of each of the figures in Problem 1. Note that a figure whose order of symmetry is 1 does not have rotational symmetry.

Challenge Problem

3. You are told that a rigid motion T has at least one fixed point. You are also told that T is direct. Show that T must be a rotation.

6-9 REFLECTIONS.

Let a line L be given.

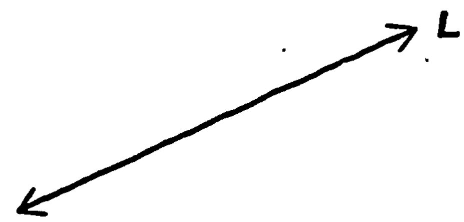

Fig. 78.

For any point P, we carry out the following construction. Through P we draw a line perpendicular to L. On this perpendicular we take a point P' so that P' is on the opposite side of L from P, and P' is at the same distance from L as P. (In case P is on L, we take P' to be P itself).

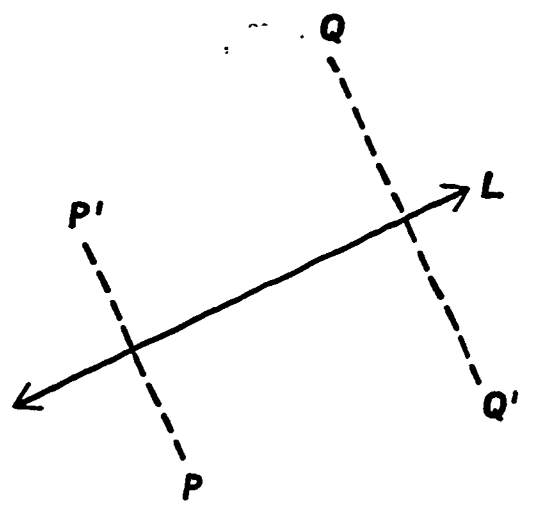

Fig. 79.

> DEFINITION 6-4. If P' is obtained from P as in the above construction, we say that P' is the result of *reflecting P in L.*

Let a line L be given in the plane. The construction above can be carried out for any point P. Hence we have a mapping T which we get by taking $T(P)$, for any point P, to be the result of reflecting P in L.

Must the mapping T, which we get in this way, be a rigid motion? If we can show that T is isometric, then, by our work in Section 6-3, we will know that T is a rigid motion. We can show that T is isometric as follows.

Take any points P_1 and P_2. The construction of P_1' and P_2' gives Figure 80 below.

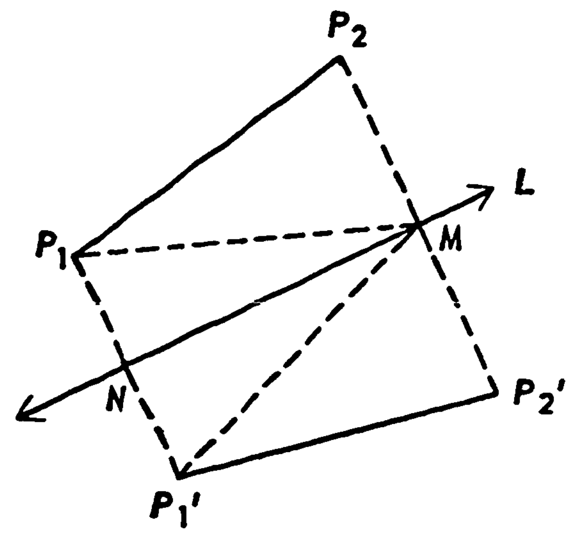

Fig. 80.

To show that T is isometric, we must show that $P_1P_2 = P_1'P_2'$.

In the above figure, $P_1N = P_1'N$ by the construction. Hence ΔP_1NM is congruent to $\Delta P_1'NM$ by SAS. Hence $P_1M = P_1'M$ and $m(P_1\widehat{M}N) = m(P_1'\widehat{M}N)$. Thus $m(P_1\widehat{M}P_2) = m(P_1'\widehat{M}P_2')$. Since $P_2M = P_2'M$ by the construction, we have ΔP_1MP_2 congruent to $\Delta P_1'MP_2'$ by SAS, and $P_1P_2 = P_1'P_2'$. (In the figure we have assumed that P_1 and P_2 lie on the same side of L. The proof is similar in case P_1 and P_2 lie on opposite sides of L.)

Hence T is isometric, and we see that T is a rigid motion. Is the rigid motion T direct or reversing? The following figure makes it clear that T must be reversing.

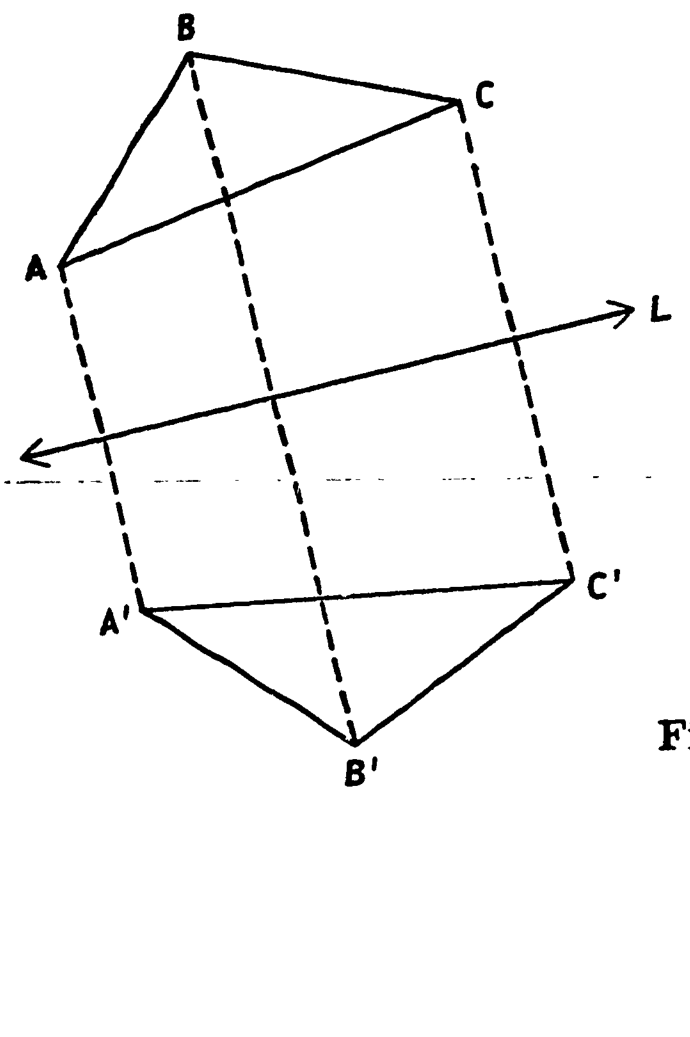

Fig. 81.

What kind of rigid motion is T? Clearly, T is what we called, in Section 6-5, a reflection, and every reflection can be obtained by using a suitable line L in the above construction for T. The line L is simply the line which remains fixed after we have turned the tracing paper over.

The rigid motions which we call reflections are closely related to the idea of reflection in a mirror. You have no doubt had the experience of standing in front of a mirror and observing your image. The image looks exactly like you and appears to be situated as far behind the mirror as you are standing in front of it. One big difference between you and your image, however, is that your left side appears as the right side of the image and vice versa. If you have been studying General Science or Physics in your course, you would have come across very many interesting experiments and examples dealing with reflection.

If T is one of the rigid motions in the plane which we are calling *reflections*, then the line L is much like the mirror-image that you would seem to see if you were to look at P in a mirror. The fact that T is reversing is like the fact that left and right are interchanged in a mirror-image.

PROBLEMS 6-9A

1. The line m is parallel to a reflecting line L. Is its image m' going to be a parallel to L also?

2. If $\overrightarrow{AB}$ is parallel to the reflecting line L, in what direction will the image $\overrightarrow{A'B'}$ point?

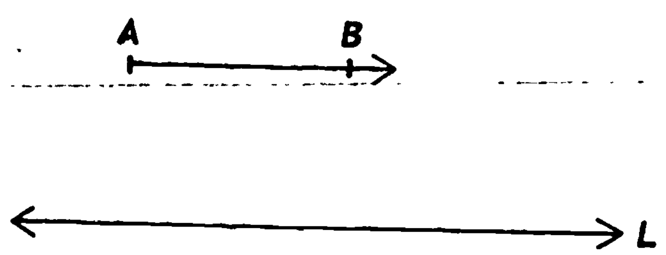

Fig. 82.

3. The line $\overleftrightarrow{AB}$ is perpendicular at O to the reflecting line L. Draw the image $\overline{A'B'}$ of $\overline{AB}$.

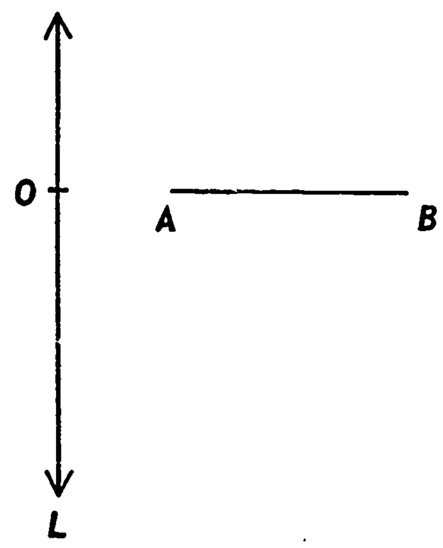

Fig. 83.

4. If, in Problem 3, one moves from left to right in going from A to B, in what direction would he move in going from A' to B'?

5. Is the result in Problem 4 similar to that obtained for a half-turn about O?

6. Draw the image of $\triangle ABC$ under a reflection along L. Is $\triangle ABC \equiv \triangle A'B'C'$, where $\triangle A'B'C'$ is the image of $\triangle ABC$? How do you know this?

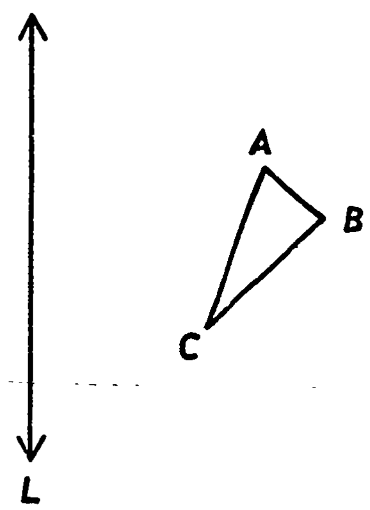

Fig. 84.

7. Draw the image of $ABCD$ under a reflection along L. Is this image congruent to $ABCD$?

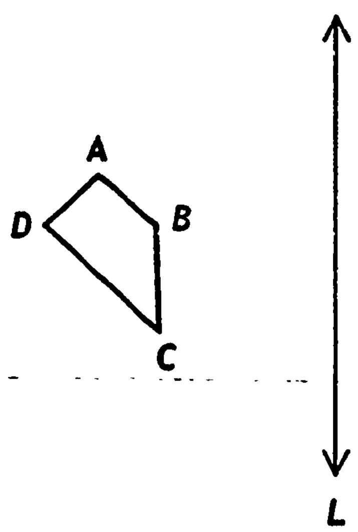

Fig. 85.

8. What will the image of a circle be under reflection on a line? The arrow in the circle is clockwise. In what direction will the arrow point in the image?

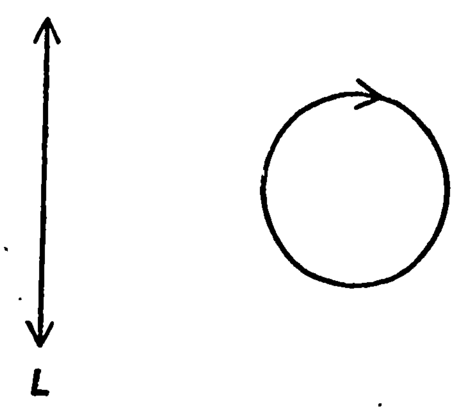

Fig. 86.

9. If $\overline{A'B'}$ is the image of $\overline{AB}$ under a line reflection, where will the line of reflection lie in the following diagrams?

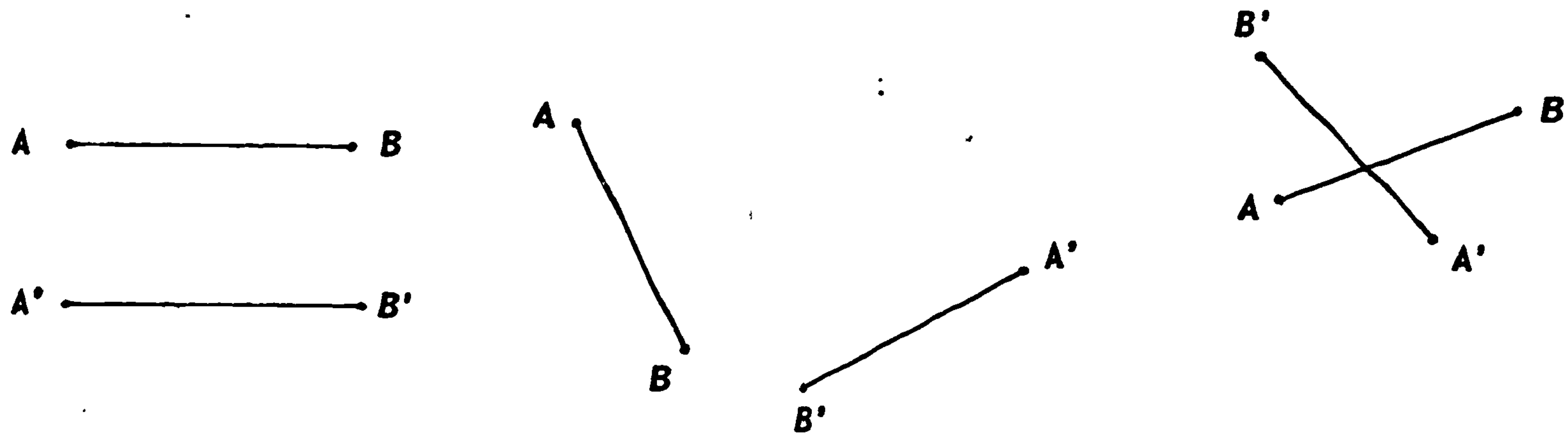

Fig. 87.

10. If $\triangle A'B'C'$ is the image of $\triangle ABC$ under a line reflection, where will the line of reflection be?

Fig. 88.

11. Is it possible to get a line of reflection which will make $\triangle A'B'C'$ an image of $\triangle ABC$ in Figure 89? If not, could you find a reason?

Fig. 89.

AXIS OF SYMMETRY

Consider the isosceles triangle ABC with $\overline{AB} \equiv \overline{AC}$ and $\overline{AD}$ an altitude. A reflection of this triangle along $\overleftrightarrow{AD}$ will give a triangle $AC'B'$ which is the same as triangle ABC with C' falling on B and B' falling on C, since the reflection of A in $\overleftrightarrow{AD}$ still gives A.

We see that, in the above reflection, each point P of the triangle is reflected to give an image point P' which is on the triangle. That is, every point P and its image belong to the triangle.

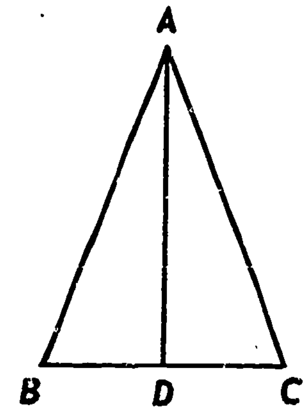

Fig. 90.

> DEFINITION 6-5. If for every point P of a figure its image P' under a reflection on a line L is also a point of the figure, the line L is an *axis of symmetry* of the figure.

REFLECTION ALONG COORDINATE AXES

A point R at a distance r along the positive x-axis becomes point R' at a distance r along the negative x-axis. Similarly a point Q at a distance q along the positive y-axis will become point Q' at a distance q along the negative y-axis.

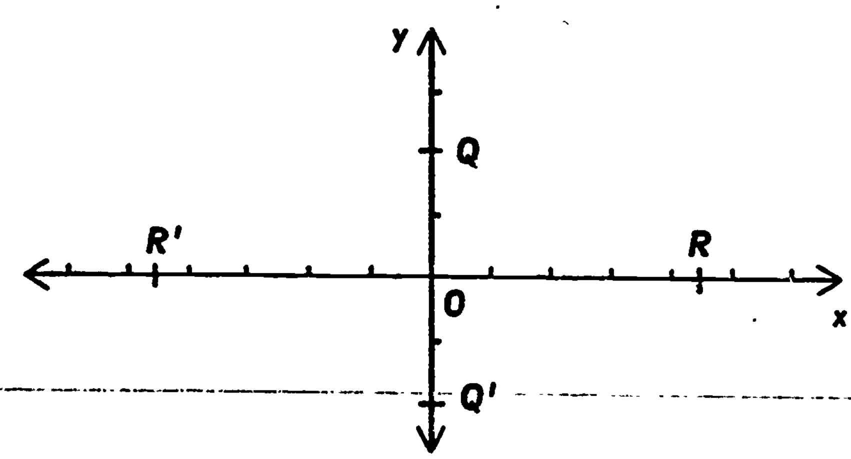

Fig. 91.

A point $P(a, b)$ when reflected in the x-axis becomes point P' $(a, -b)$. A point $P(a, b)$ when reflected in the y-axis becomes point P'' $(-a, b)$.

In general, the point (x, y) becomes $(x, -y)$ under a reflection in the x-axis and the point (x, y) becomes $(-x, y)$ under a reflection in the y-axis.

PROBLEMS 6-9B

1. Lines L and m are the bisectors of the angles formed by the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ in Figure 92 below. Show that L is an axis of symmetry of the figure.

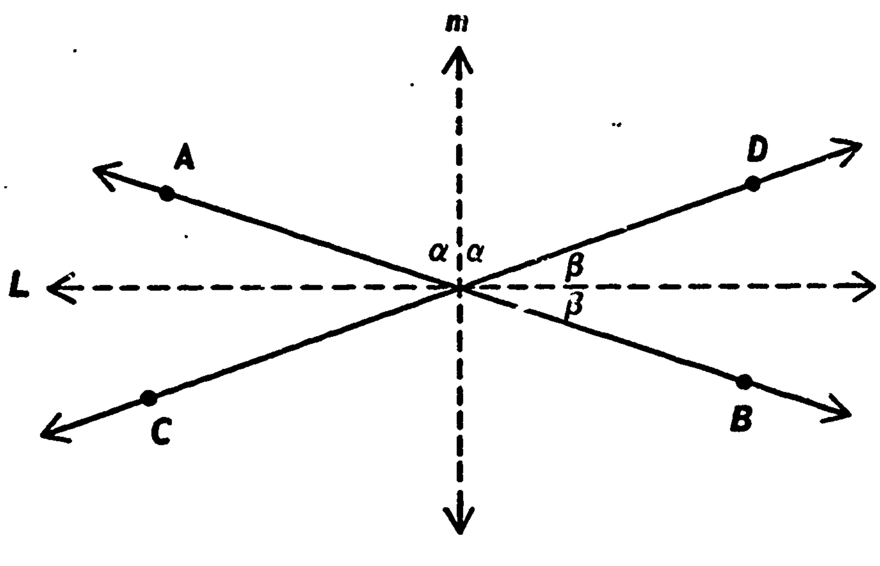

Fig. 92.

Show that L and m are the locus of points equidistant from the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

2. We say that a point P is *reflected in a point* O, if its image P' is such that $\overleftrightarrow{POP'}$ is a straight line and $PO = OP'$. Show that this is the same as doing a half turn about O.

Challenge Problem

3. What will the coordinates of the point (a, b) become under a reflection along the line $y = x$?

6-10 ONE MOTION FOLLOWED BY ANOTHER.

Let a rigid motion R and a rigid motion S be given. For example, R might be the rotation about O which carries ΔABC to $\Delta A'B'C'$ in the following figure, and S might be the translation given by the vector $\mathbf{U}$ in the figure.

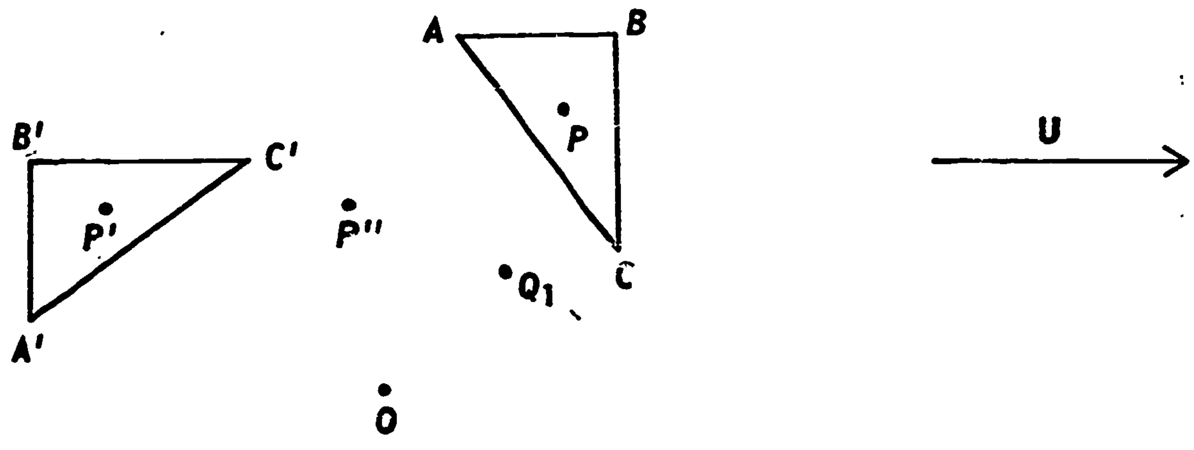

Fig. 93.

Now for any point P, we carry out the following construction. First we find the point $R(P)$, which we call P'. Then we take P' and find the point $S(P')$, which we call P''. We can carry out this construction for any point P. Hence we have a mapping T which we get by taking $T(P)$, for any point P, to be the point P'' obtained from P by this construction.

Must the mapping T, which we get in this way, be a rigid motion? If we can show that T is isometric, then, by our work in Sections 6-3 and 6-6, we will know that T is a rigid motion. Take any points P_1 and P_2. To show that T is isometric, we must show that $P_1P_2 = P_1''P_2''$. But $P_1P_2 = P_1'P_2'$, since R is isometric, and $P_1'P_2' = P_1''P_2''$, since S is isometric. Hence $P_1P_2 = P_1''P_2''$, and we have that T is isometric. Therefore T must be a rigid motion.

> DEFINITION 6-6. Given any two rigid motions R and S, the new rigid motion T which we get when R is followed by S is called the *combination of R with S*. We shall also refer to is as simply "R followed by S".

Think for a moment of the tracing paper idea of a rigid motion. R results from a certain movement of the tracing paper and S results from a certain movement of the tracing paper. If we make the first of these movements and then follow it with the second, we reach a final position (after the two movements) which could have been reached by a single movement of the tracing paper. T is the rigid motion which results from this single movement.

What kind of rigid motion is T? This will depend, of course, on the rigid motions R and S. In Figure 93 above, T is a rotation about the point marked Q_1. (You can check this with some tracing paper.)

What would happen if we carried out the two motions R and S in reverse order? Then the figure would be as follows.

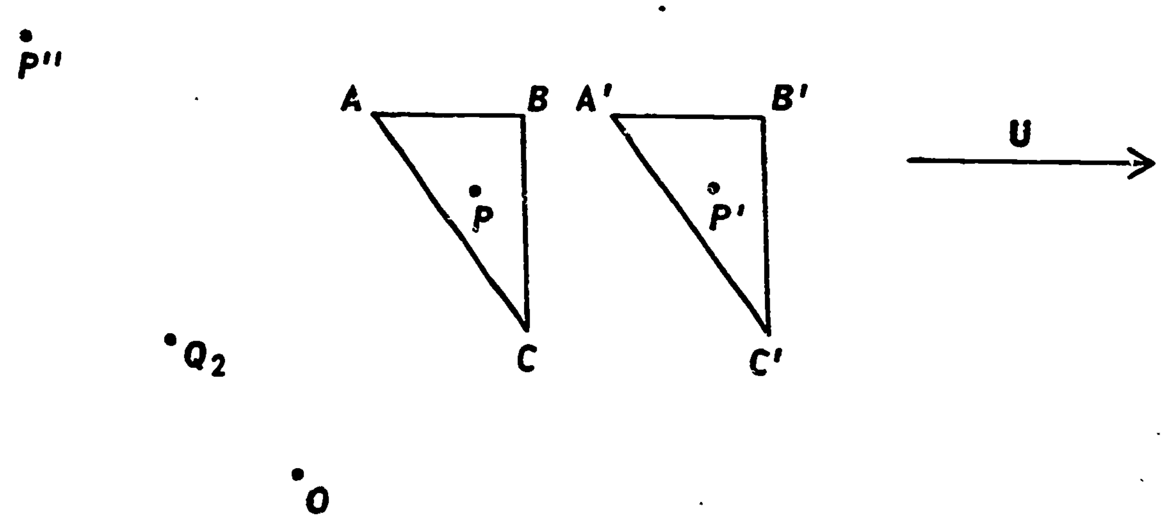

Fig. 94.

The resulting rigid motion, as you can check with tracing paper, is a rotation about the point Q_2. Clearly we get two different motions, depending upon the order in which we take R and S.

This example tells us an important fact about combinations of rigid motions: if R and S are *rigid motions*, *R followed by S need not be the same as S followed by R*.

In special cases, R followed by S and S followed by R may be the same. For example, if R and S are the translations given by **U** and **V** in the following figure, you can check that, in this case, R followed by S is the same as S followed by R. Each is the transformation T given by **W**.

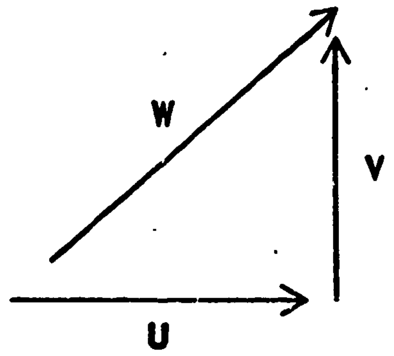

Fig. 95.

There are some general rules which we can use to tell what kind of rigid motion T is if we already know what kinds of motion R and S are. We now look at some of these rules.

TRANSLATION FOLLOWED BY TRANSLATION

We remind ourselves that in a translation every point in a line or figure moves the same distance and in the same direction, and that a translation can be described by means of a vector whose length gives the distance that every point moves and whose direction gives the direction that every point moves.

Fig. 96.

In Figure 96 the triangle A can be moved to A' by the translation given by vector **U** and A' moved to A'' by the translation given by vector **V**. Hence A is moved to A'' by translating first by **U** and then by **V**. But clearly A is also moved to A'' by the translation given by **W**. The three vectors are drawn to the right in Figure 96.

This example suggests that a translation followed by another translation must always be a translation. We can state this precisely as follows:

A translation R followed by a translation S equals a translation T. If R is given by a vector **R** and S is given by a vector **S** whose initial point is the end point of **R** (see Fig. 97) then T is given by vector **T**, with initial point at the initial point of **R** and end point at the end point of **S**.

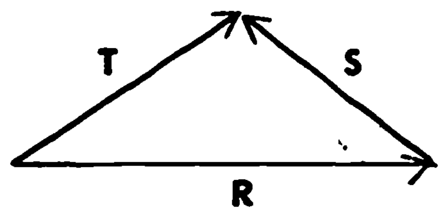

Fig. 97.

Try several pairs of translations like R and S on different figures to test the correctness of the above statement.

PROBLEMS 6-10A

1. In Figure 96 above, find two other translations besides **U** and **V** which have the same effect as **W** when used one after the other.

2. Draw a triangle A and two vectors **U** and **V**. Draw the image of A under **U**; i.e., draw A' where A is moved by **U**. Draw A'' where A' is moved by **V**. Draw a vector showing a single translation moving A to A''. Is this equal to **U** followed by **V**?

3. Draw three vectors **S**, **T**, **U** in any position. Draw a vector **V** showing a translation equal to translation **S** followed by translation **T**, followed by translation **U**.

4. Draw a vector **V** showing a translation. Draw two vectors **S** and **T** at right angles showing translations such that **V** equals **S** followed by **T**.

5. If A is moved to A' by translation T, where does T move A'? If A' is moved to A'' by T, where does T move A''? Draw a diagram showing A, A', A'', A''', A^{iv} where T moves each point to the next one to its right.

EXAMPLE

In Figure 98, the line segment $\overline{CD}$ can be moved to $\overline{C'D'}$ by a half-turn about H followed by a translation W. $\overline{CD}$ can also be moved to $\overline{C'D'}$ by a half-turn about K. Any other figure will be moved to the same position by the half-turn about K as by the half-turn about H followed by translation W. We say the half-turn about K equals the half-turn about H followed by the translation W.

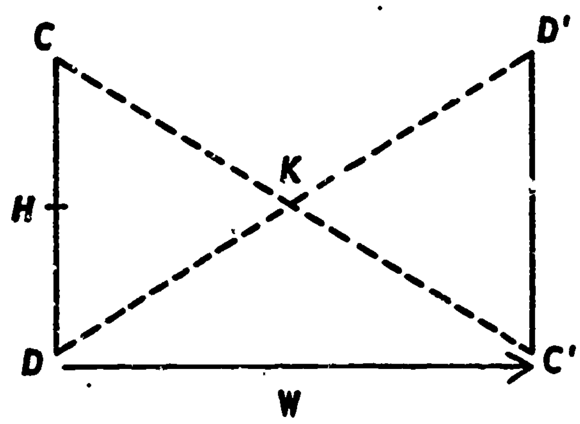

Fig. 98.

PROBLEMS 6-10B

1. In Figure 98, suppose $\overline{CD}$ is moved by translation W to a new position, say $\overline{C''D''}$. Can you find a half-turn which will move $\overline{C''D''}$ into $\overline{CD}$?

2. In Figure 98, can you find a half-turn about another point and a translation to follow it which is equal to a half-turn about K; i.e., moves $\overline{CD}$ to $\overline{C'D'}$?

3. Suppose $\overline{C''D''}$ in Problem 1 is moved by the half-turn at H; i.e., $\overline{CD}$ is moved by W followed by a half-turn at H. Where is the final position of $\overline{CD}$? Is it the same as $\overline{CD}$?

ROTATION FOLLOWED BY ROTATION

Consider a point P rotated about a point O through angle α to give P' followed by another rotation of P' through angle β to give P''. Clearly a rotation of P through angle $\alpha + \beta$ will also give P''.

However if the rotations are about two different centres the result is not so simple to obtain. Here we shall only consider combinations of some special rotations—such as half-turns—about different centres. We give two facts about such combinations.

(1) A half-turn about point H followed by another half-turn about H is the identity.

Can you show this?

(2) The result of a half-turn about a point H followed by a half-turn about a point K is a translation T, shown by a vector twice the vector from H to K.

To show this, let P be moved to P' by H and let P' be moved to P'' by K in Figure 99. Since H, K are the mid-points of the sides, $\overline{P'P}$ and $\overline{P'P''}$ of the triangle $PP'P''$, then $PP'' = 2HK$.

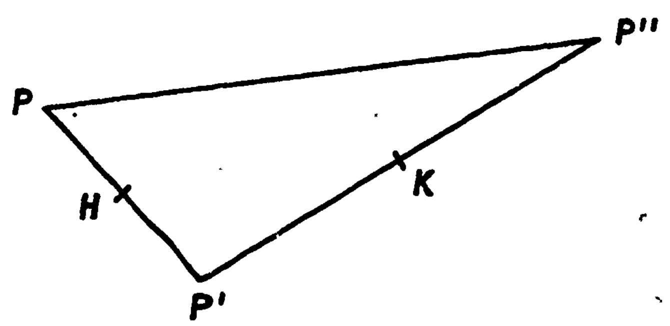

Fig. 99.

PROBLEMS 6-10C

1. Mark two points H, K about one inch apart as centres of half-turns. Take any three points P, Q, R and construct their images under the half-turn about H followed by the half-turn about K; i.e., find their images P', Q', R' respectively under the half-turn about H and then the images of these points P', Q', R' under the half-turn about K, which could be labelled P'', Q'', R'' respectively. What can you say about the length of the segments $\overline{PP''}$, $\overline{QQ''}$, $\overline{RR''}$? Are these segments parallel to each other?

2. Let S be a translation shown by a vector LL'. Let F and G be any two points whose distance apart is half the distance LL', and $\overline{FG}$ is parallel to $\overline{LL'}$. Show that the translation $S = GF$, where GF means performing the half-turn about F followed by the half-turn about G.

3. $ABCD$ are the vertices of a parallelogram. Show that the half-turn about A followed by the half-turn about B is the same motion as the half-turn about D followed by the half-turn about C.

4. Let KH be a half-turn about H followed by a half-turn about K. What would you mean by HK? Is $HK = KH$? If not, what is the difference?

5. Let H, J, K be three half-turns about points which lie on a line, equally spaced with J between H and K. Show that $KJH = J$. (How would you define KJH?)

6. P, Q, R, S are half-turns about points on a line shown in the diagram with $\overline{PQ} = \overline{RS}$.

   Show that $SQP = R$.

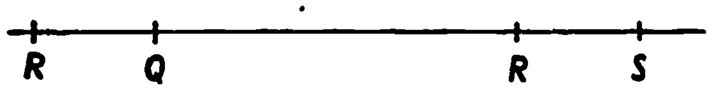

Fig. 100.

7. Let A, B, C, D be four half-turns about successive vertices of a parallelogram. Show that $CBA = D$ and that $ABCD$ is the identity.

REFLECTION FOLLOWED BY REFLECTION

We shall get information about a reflection followed by a reflection from the following problems.

PROBLEMS 6-10D

Consider two parallel reflecting lines L and m, with a point P first reflected along L to give P' and P' then reflected along m to give P''.

1. Suppose the distance of P from L is p, and the distance between L and m is d. What is the length of $\overline{PP''}$?

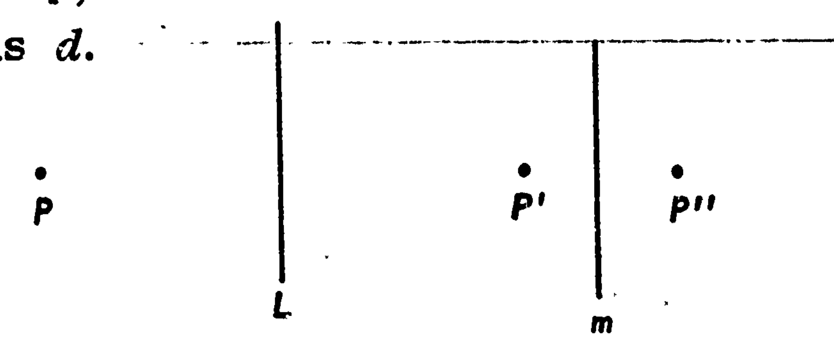

Fig. 101.

2. Use the same values as in Problem 1, but now P is reflected first along m to give P_1' and then P_1' is reflected along L to give P_1''. What is the length of $\overline{PP_1''}$?

3. Consider the line segment $\overline{PQ}$ in Figure 102. Can you determine the image $\overline{P''Q''}$ which results from reflecting first along L and then along m? Can you show that this double reflection is equivalent to a translation?

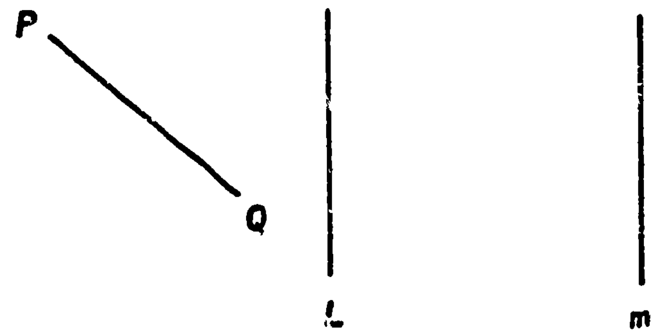

Fig. 102.

4. Replace the numbers 1 and 2 in Figure 103 by the image of B under reflection on f and the image of B under reflection on f, followed by reflection on g.

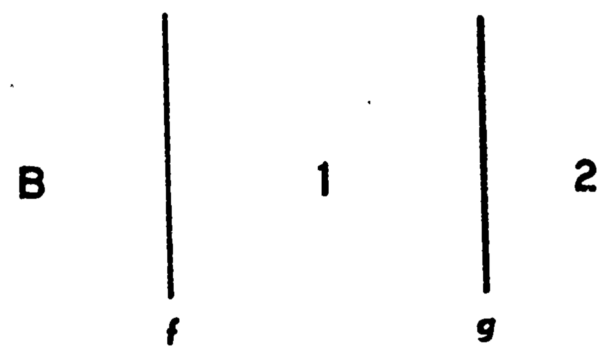

Fig. 103.

5. Let f and g be parallel lines three units apart. Find the second images of the following points when reflected first in f, then in g, where f is to the left of g.
   (a) P, one unit to the right of f.
   (b) Q, one unit to the left of f.
   (c) R, four units to the left of f.
   (d) S, one unit to the right of g.

6. Find the second images of the above points if they are reflected first in g, then in f.

We see from the solutions to the above problems that the result of two reflections along parallel lines is a translation shown by a vector perpendicular to the lines of reflections of length twice the distance between the parallel lines.

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7. Let T be a translation shown by the vector **T**. Find two parallel lines of reflection f and g, so that f followed by g equals the translation T.

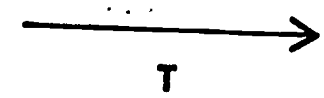

Fig. 104.

A BASIC THEOREM

Let R and S be rigid motions, and let T be the rigid motion R followed by S. We looked at some rules above for finding out what kind of motion T is, if we know what kinds of motion R and S are. We now look at several further facts that can be useful in getting information about T. These facts will not always tell us exactly what T is, but they can give some helpful information. The first of these facts is the following.

> If R and S are both direct, then T must be direct. If R and S are both reversing, then T must be direct. If R is direct and S is reversing, or if R is reversing and S is direct, then T must be reversing.

We see from the definition of direct and reversing why this fact is true. For example, if R requires that the tracing paper be turned over and S requires that the paper be turned over again, then clearly, in the resulting rigid motion T, the tracing paper has the same side up that it had to start with.

The following basic theorem gives an important and useful fact about direct rigid motions, as we shall see.

> THEOREM 6-1. Let T be a rigid motion. If T is direct, then either T is a translation or T is a rotation.

We shall not give the proof of this theorem here. Instead, we shall tell you how to find out exactly what kind of rigid motion T is, if you know that T is direct, and if you are given two triangles ABC and $A'B'C'$ such that $A' = T(A)$, $B' = T(B)$, and $C' = T(C)$.

Do the following:

Draw the segments $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. At least two of these segments have positive length (otherwise the triangles are the same). Draw the perpendicular bisector line for each of the segments with positive length. At least two of these perpendicular bisector lines will be distinct. (Can you prove this?) We show some of the possibilities in the following figures. (In these figures, the perpendicular bisector lines are marked L_1, L_2, and L_3.)

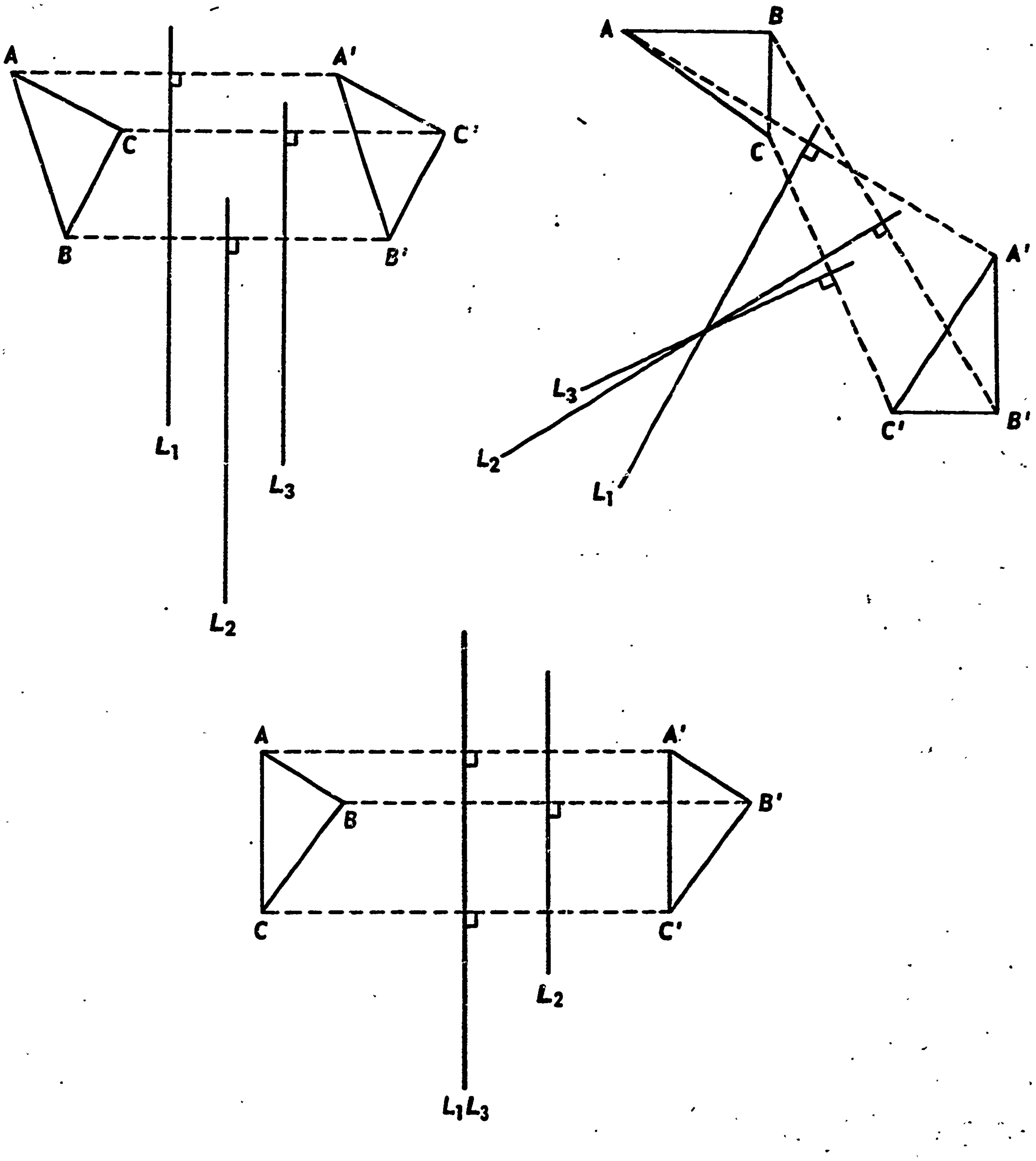

Fig. 105.

Choose two distinct perpendicular bisector lines. If these lines are parallel, then you have T as the translation which moves every point parallel to $\overline{AA'}$ through a distance AA'. If these lines intersect (call this point of intersection O), then you have T as the rotation about O which carries A to A'.

The method will always work. We do not prove this here. If you can prove that the method always works, you will be very close to having a proof for Theorem 6-1.

Theorem 6-1 is a strong tool for getting information about combinations of rigid motions. Look at the following example. In Figure 106, let R be a reflection about L_1 and let S be a reflection about L_2. Let T be R followed by S.

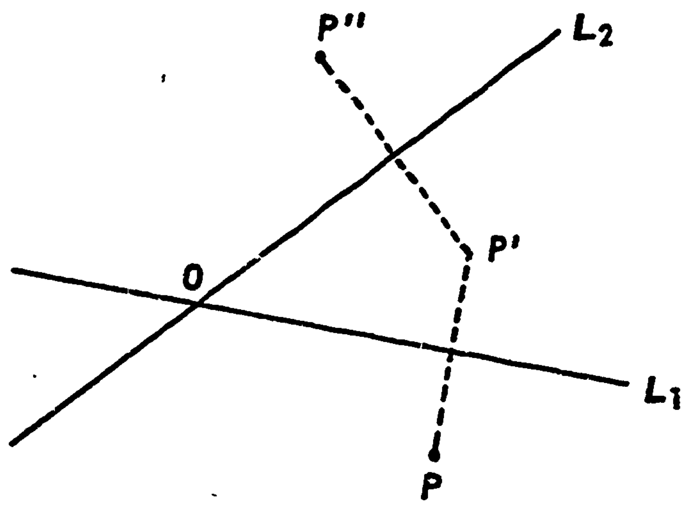

Fig. 106.

What kind of motion is T? We can find this very simply as follows. Since R and S are each reversing, we know that T must be direct. Hence, by Theorem 6-1, T must be either a translation or a rotation. But the point O is a fixed point of T. Hence T must be a rotation about O. (It cannot be a translation, since no translation, except the identity motion, can have a fixed point. T is not the identity motion, as we see from the points P, P', and P'' in the figure.) To find the angle of rotation, we need only measure the angle $P\hat{O}P''$ in the figure.

A second basic theorem can be given for rigid motions that are reversing. We state it below as Theorem 6-2, but do not consider it further.

> DEFINITION 6-7. T is called a *glide-reflection*, if T is R followed by S, where R is a reflection and S is a translation given by a vector parallel to the line of reflection of R.

For example, if R is reflection in L, and S is the translation given by **U** in the following figure, then the motion R followed by S, which we are calling a glide-reflection, is the motion which takes $\triangle ABC$ to $\triangle A''B''C''$.

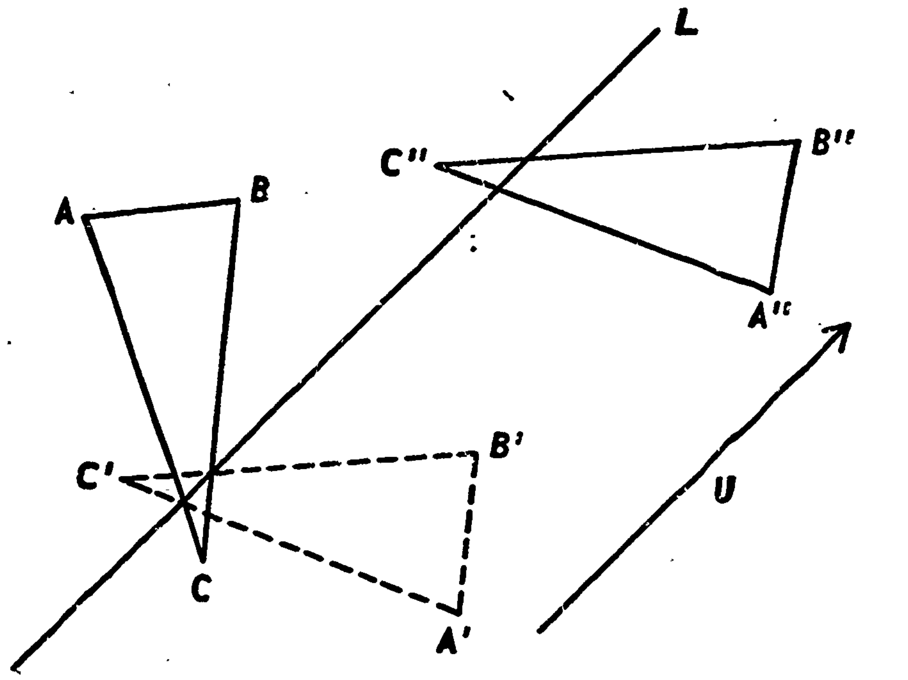

Fig. 107.

> THEOREM 6-2. Let T be a rigid motion. If T is reversing, then either T is a reflection or T is a glide-reflection.

Hence we know from these two theorems that *every* rigid motion is either a translation, a rotation, a reflection, or a glide-reflection.

6-11 MORE EXAMPLES OF THE USE OF MOTIONS.

In Section 6-4 we looked at two examples of the use of rigid motions to solve problems. Both of these examples used translations. We now look at several more examples. These new examples will use rotations and reflections as well as translations.

EXAMPLE 1. Take any triangle ABC. Construct equilateral triangles on $\overline{AB}$ and $\overline{BC}$ as in the figure.

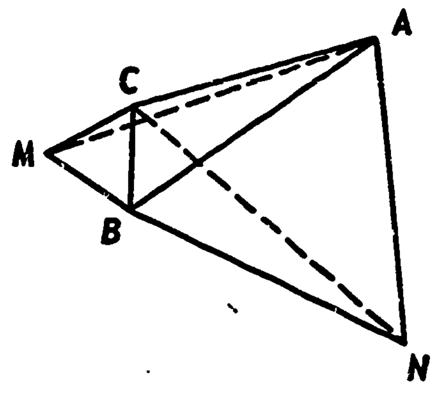

Fig. 108.

We wish to show that $AM = CN$.

A congruent triangle proof can be used to show this (by using SAS to get $\triangle ABM$ congruent to $\triangle NBC$). But the idea of a rotation can be used to get an even quicker proof as follows. *Let T be a rotation about B through an angle of 60^0.* Clearly, $T(N) = A$ (since $BN = BA$ and $m(N\hat{B}A) = 60^0$), and $T(C) = M$ (since $BC = BM$ and $m(C\hat{B}M) = 60^0$). Thus segment $\overline{AM}$ is the image of segment $\overline{CN}$ under T, and $AM = CN$ by the isometric property of rigid motions.

PROBLEMS 6-11A

1. In the figure for Example 1, let Q be the point of intersection of $\overline{AM}$ and $\overline{CN}$. What is the measure of $N\hat{Q}A$?

2. In the following figure, ABC is any triangle, and squares have been constructed on $\overline{AB}$ and $\overline{BC}$.

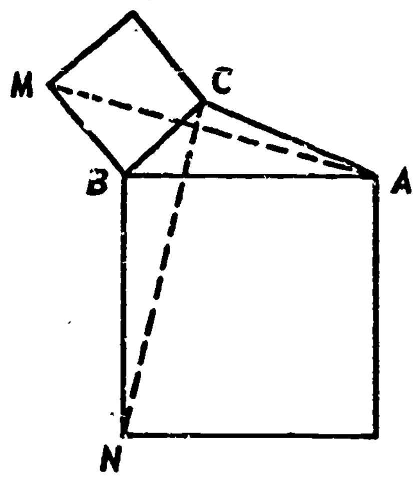

Fig. 109.

Use a rotation to show that $AM = CN$.

3. In Problem 2, show that $\overline{AM}$ is perpendicular to $\overline{CN}$.

EXAMPLE 2. Let ABC be a triangle (Figure 110) in which $m(B\hat{A}C) = m(B\hat{C}A)$.

It is a fact of elementary geometry that AE must be equal to BC. Let us assume, for the moment, that you have forgotten this fact, but that you do remember the basic properties of rigid motions. (These were listed as Properties (1) to (6) in Section 6-3). How could you use what you do know about rigid motions to prove that $AB = BC$?

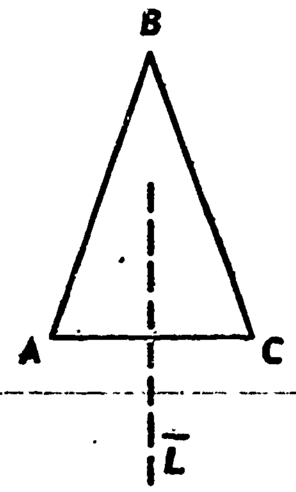

Fig. 110.

One very simple proof would be as follows. Take the perpendicular bisector of $\overline{AC}$, call it $\overline{L}$. Let T be the rigid motion: *reflection in* $\overline{L}$. Since T leaves the measure of an angle unchanged (Property (6) in Section 6-3), the image of $\overline{AC}$ under T must also be perpendicular to $\overline{L}$. Hence, since $\overline{L}$ is a bisector, the image of A is C and the image of C is A. Since angles $B\widehat{A}C$ and $B\widehat{C}A$ have equal measure, the image of angle $B\widehat{A}C$ must be angle $B\widehat{C}A$, and the image of angle $B\widehat{C}A$ must be angle $B\widehat{A}C$. Hence the image of line $\overleftrightarrow{AB}$ is line $\overleftrightarrow{CB}$ and the image of line $\overleftrightarrow{CB}$ is line $\overleftrightarrow{AB}$. Since point B lies on both $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$, the image of B must lie on both $\overleftrightarrow{CB}$ and $\overleftrightarrow{AB}$. But this means that the image of B has to be B itself. Thus the image of segment $\overline{AB}$ is segment $\overline{CB}$, and the image of segment $\overline{CB}$ is segment $\overline{AB}$. By the isometric property, we have $AB = CB$, which is what we wanted.

PROBLEMS 6-11B

1. Let ABC be a triangle in which $AB = CB$. It is a fact of elementary geometry that $m(C\widehat{A}B)$ must be equal to $m(A\widehat{C}B)$. Assume, for the moment, that you have forgotten this fact, but that you do remember the basic properties of rigid motions. Use a rigid motion to prove that $m(C\widehat{A}B) = m(A\widehat{C}B)$. (Hint: let L be the bisector of $A\widehat{B}C$ in Figure 111, and make a reflection in L.)

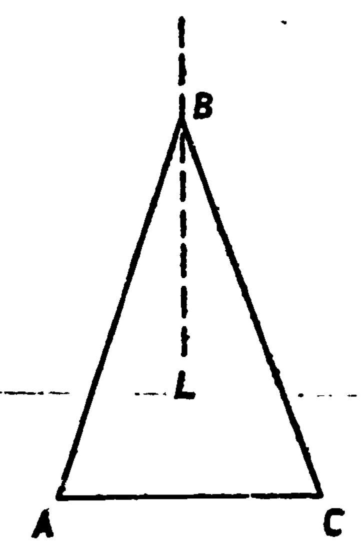

Fig. 111.

Note: As we saw in *Secondary Three,* it is possible to get the facts of geometry from a certain set of *axioms* about points, lines, angles, betweenness, and congruence. It is also possible to get the facts of geometry in an entirely different way by taking the idea of rigid motion as basic (along with the ideas of point, line, and angle), and by starting with a set of axioms which includes basic facts about rigid motions (such as those listed in Section 6-3). This set of axioms is quite different from the set of axioms used in *Secondary Three*. In this new deductive theory, we can get all the same facts of geometry that we got before, but the proofs are quite different, and the theorems appear in a different order from before. Example 2 above and the problem which follows it shows us what such proofs would be like. We do not consider this new theory any further, and we do not give the set of axioms for it here.

In the next two examples, we shall look at more difficult problems. In each of these problems, rigid motions can be used to get a solution which is shorter than any other solution known. In solving each problem, we will allow ourselves to use facts that we know about rigid motions together with any other facts of geometry that are already known to us.

EXAMPLE 3.

Let $ABCD$ be any convex quadrilateral. Construct equilateral triangles APB, BQC, CRD, and DSA as in Figure 112. Note that two triangles are constructed towards the inside of the quadrilateral and two are constructed towards the outside.

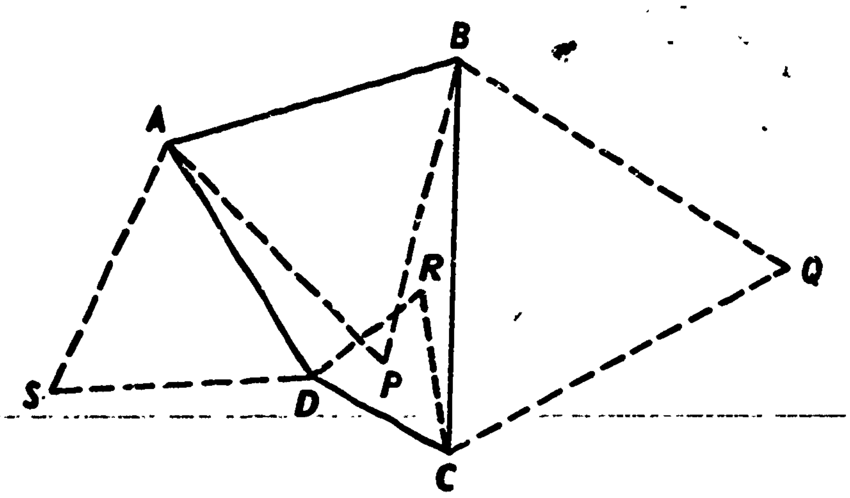

Fig. 112.

We wish to show that the points R, Q, P, and S must form the vertices of a parallelogram. We use rigid motions to prove this as follows.

Let T_1 be a rotation through 60^0 about A, and let T_2 be a rotation through -60^0 about C. Let U be the rigid motion: T_1 followed by T_2. Since T_1 and T_2 are direct, U must be direct. Hence by Theorem 6-1 of Section 6-10, U must be a translation or a rotation. If it is a rotation through an angle β, then, by our work at the end of Section 6-8, every line makes an angle β with its own image. Let L be any line. Let L_1 be the image of L under T_1 and let L_2 be the image of L_1 under T_2. Then L_1 forms an angle of 60^0 with L, and L_2 forms an angle of -60^0 with L_1. Hence L_2 forms an angle of 0^0 with L. *Hence, if* U *is a rotation, it can only be a rotation through* 0^0*, which is the identity motion.* Thus U must be either the identity motion or a translation.

Let us see where U carries the points S and P. T_1 carries S to D and T_2 carries D to R. Hence $U(S) = R$. Also, T_1 carries P to B, and T_2 carries B to Q. Hence $U(P) = Q$. Thus U is a translation which carries segment $\overline{SP}$ to segment $\overline{RQ}$.

By the isometric property of U, $SP = RQ$. Since every translation carries a line parallel to itself (or else onto itself), $\overleftrightarrow{SP}$ is parallel to $\overleftrightarrow{RQ}$. Since the quadrilateral $SQPR$ has a pair of opposite sides which are equal and parallel, it must be a parallelogram. This is what we set out to show.

EXAMPLE 4. In the following figure, let ABC be any triangle. ABM, BCN and CAO are equilateral triangles constructed on the sides of triangle ABC. Let P, Q, R be the centres of these three equilateral triangles. (By *centre* of an equilateral triangle we mean the point at which the three perpendicular bisectors of the sides meet. This is the point about which an equilateral triangle has rotational symmetry.)

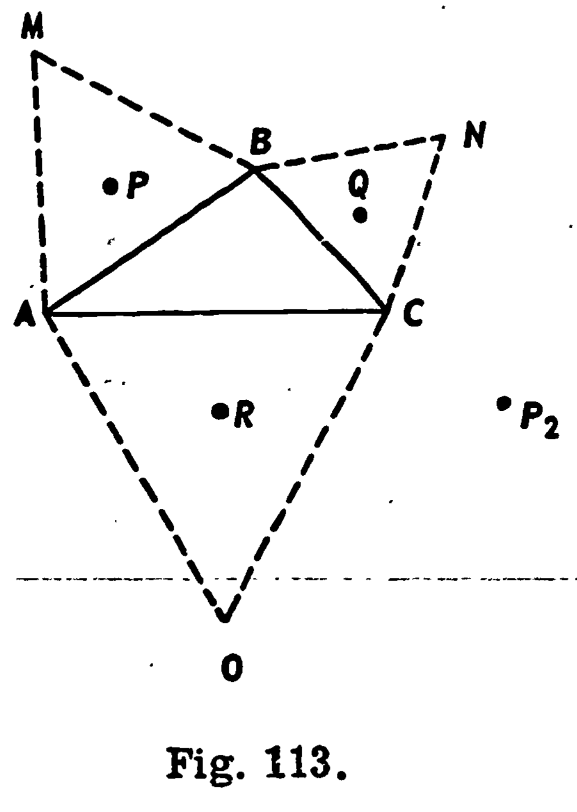

Fig. 113.

We wish to show that P, Q, and R must form the vertices of an equilateral triangle.

Let T_1 be a rotation through 120^0 about P. Let T_2 be a rotation through 120^0 about Q, and let T_3 be a rotation through 120^0 about R. Let U be the rigid motion: T_1 followed by T_2 followed by T_3. U is the result of three rotations, each of 120^0. Hence, in the same way as in Example 3, we see that U is either a translation or a rotation through 360^0. But this means that U must be a translation or the identity motion.

From the figure, we see that $T_1(A) = B$, $T_2(B) = C$, and $T_3(C) = A$. Thus $U(A) = A$, and A is a fixed point of U. But the only translation which has a fixed point is the identity motion. Hence U must be the identity motion.

Now take the point P. Let P_1 be the image of P under T_1, let P_2 be the image of P_1 under T_2, and let P_3 be the image of P_2 under T_3. Since T_1 is a rotation about P, we see that P_1 is the same as P. Since U is the identity motion, we see that P_3 is the same as P. Hence P, Q, P_2 and R form the following figure.

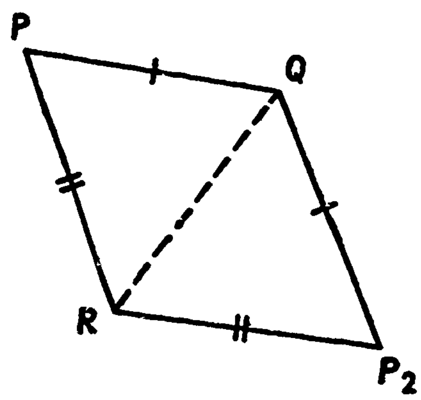

Fig. 114.

In this figure, $PQ = P_2Q$ by the isometric property of T_2, and $P_2R = PR$ by the isometric property of T_3. Hence, by SSS, triangles PQR and P_2QR are congruent. Hence $m(P\widehat{Q}R) = m(P_2\widehat{Q}R)$ and $m(P\widehat{R}Q) = m(P_2\widehat{R}Q)$. But $m(P\widehat{Q}P_2) = 120^0$ since T_2 is a rotation through 120^0, and $m(P\widehat{R}P_2) = 120^0$ since T_3 is a rotation through 120^0. Hence $m(P\widehat{Q}R) = \frac{1}{2} m(P\widehat{Q}P_2) = 60^0$, and $m(P\widehat{R}Q) = \frac{1}{2} m(P\widehat{R}P_2) = 60^0$. But any triangle with two angles of 60^0 must be equilateral. This is what we set out to prove.

*APPENDIX TO CHAPTER 6

In Section 6-3, we listed six properties of rigid motion, and we said that we could prove Properties (2), (3), (4), (5), and (6) from Property (1) (the isometric property) without using the idea of tracing paper at all. We give these proofs here.

Let T be a mapping from the plane to the plane which has Property (1).

Proof of Property (2). If P and Q are distinct, then $Q > O$. By Property (1), $P'Q' = PQ$. Hence $P'Q' > O$, and P' and Q' must be distinct.

Proof of Property (3). Let Q be any given point. We shall find a point P such that $T(P) = Q$. Take any triangle in the plane and let P_1, P_2, and P_3 be its vertices. Let P_1', P_2', and P_3' be the images of P_1, P_2, and P_3 under T. By Property (1), $P_1'P_2' = P_1P_2$, $P_2'P_3' = P_2P_3$, and $P_3'P_1' = P_3P_1$. Hence, by SSS, P_1', P_2' and P_3' form the vertices of a triangle (and $\triangle P_1P_2P_3 \equiv \triangle P_1'P_2'P_3'$). This means that the points P_1', P_2', P_3' do not all lie on the same straight line. Now at least one out of the following three cases must occur. Either (i) P_1 is the same point as Q, or (ii) Q does not lie on $\overleftrightarrow{P_1'P_2'}$, or (iii) Q does not lie on $\overleftrightarrow{P_1'P_3'}$.

(i) If P_1 is the same point as Q, then P_1 is the point P that we want.

(ii) If Q does not lie on $\overleftrightarrow{P_1'P_2'}$, we do the following.

With P_1 as centre, we draw a circle of radius $P_1'Q$, and with P_2 as centre, we draw a circle of radius $P_2'Q$. Let M and N be the points of intersection of these two circles. We now have a figure that looks like the following.

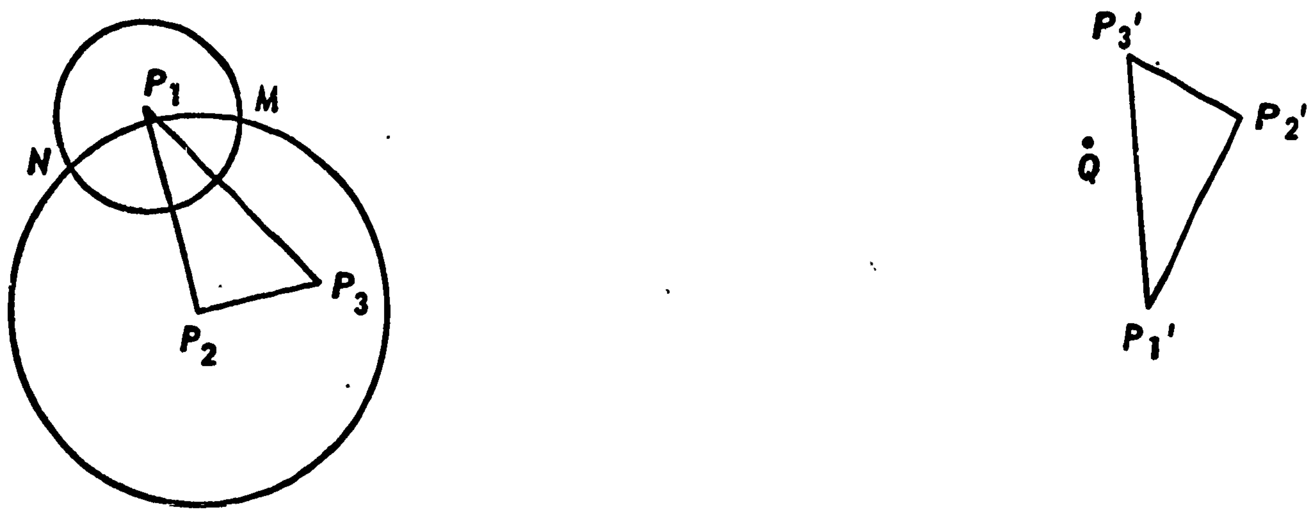

Fig. 115.

Take the image of M, which we call M', and the image of N, which we call N'. By Property (1), $P_1'M' = P_1'N' = P_1Q$, and $P_2'M' = P_2'N' = P_2'Q$. Since M and N are distinct, M' and N' must be distinct (by Property (2) which we have already proved). Hence M' and N' fall on opposite sides of $\overleftrightarrow{P_1'P_2'}$. By SSS, $\Delta P_1'P_2'Q \equiv \Delta P_1'P_2'M' \equiv \Delta P_1'P_2'N'$. Hence, if Q is on the same side of $\overleftrightarrow{P_1'P_2'}$ as M', then Q must be the same point as M', and if Q is on the opposite side of $\overleftrightarrow{P_1'P_2'}$ from M', then Q must be the same point as N'. If Q is the same as M', then M is the point P that we want; if Q is the same as N', then N is the point P that we want.

(iii) If Q does not lie on $\overleftrightarrow{P_1'P_3'}$, we carry out the same proof as for (ii), but use P_3' in place of P_2'. We again find the point P that we want.

Thus in all three cases we can get the point P that we want. This ends the proof of Property (3).

Proof of Property (4). Let L be a given straight line. Take two distinct points P_1 and P_2 on L. Let P_1' be the image of P_1 and let P_2' be the be the image of P_2. Let L' be the line determined by P_1' and P_2'. To prove (4) we need to show that every point on L has its image on L', and that every point on L' is the image of some point on L.

Let Q be any point. By Property (3), which we have already proved, there is a point P such that $Q = T(P)$. If Q is not on L' then, since the triangle $P_1P_2P_3$ is congruent to the triangle $P_1'P_2'O$, P is not on L. Conversely, if P is not on L, then, by the same congruent triangles, Q is not on L'. This gives us the result we want, because it shows that every point on L must have its image on L' and, by Property (3), it shows that every point on L' must be the image of some point on L.

Proof of Property (5). By Property (4), which we have proved, $P'\widehat{Q'}R'$ is the image of $P\widehat{Q}R$. Since, by Property (1), and SSS, $\triangle PQR \equiv \triangle P'Q'R'$, we have that $P'\widehat{Q'}R' = P\widehat{Q}R$ in measure.

Proof of Property (6). Let L_1 and L_2 be two parallel lines. Let L_1' be the image of L_1, and let L_2' be the image of L_2. By Property (4), which we have proved, L_1' and L_2' are straight lines. If L_1' and L_2' intersect at some point Q, then, by Property (3) and Property (4) there is a point P such that $T(P) = Q$, P lies on L_1, and P lies on L_2. But if P lies on L_1 and L_2, L_1 and L_2 cannot be parallel. This shows that L_1' and L_2' must be parallel.

Thus we have proved Properties (2), (3), (4), (5), and (6) from Property (1).

LIST OF THEOREMS
AND OTHER IMPORTANT RESULTS

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